Strategyproofness, Leontief Economies and the Kalai-Smorodinsky Solution

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Abstract

We present new characterizations of the Raiffa and Kalai-Smorodinsky solutions based on strategyproofness of an allocation mechanism for an underlying economy with Leontief preferences. Our first result shows that the 2-player Kalai-Smorodinsky solution is the unique bargaining solution which if Efficient, Symmetric, Scale Invariant and Strategyproof on a Leontief economy. Next we consider the class of weighted DRF mechanisms which are of great practical interest and are group strategyproof. We show that they satisfy and are characterized by a new proportional consistency axiom, Ratio Consistency, and that the multi-player Raiffa solution is the unique unique weighted DRF mechanism that generates a bargaining solution. Finally, we present a result that complements Imai's characterization of the multi-player lexicographic-Kalai-Smorodinsky solution. We show that strategyproofness, combined with other basic axioms, implies a version of the individually rational individual monotonicity which when combined with the Independence of Irrelevant Alternatives other than Ideal Point axiom characterizes the lexicographic-Kalai-Smorodinsky solution for 3-players and is conjectured to characterize it for an arbitrary number of players. These results shed new light and provide new insights into the study of mechanisms on Leontief economies and bargaining theory as well as having practical applications in the design of modern data-centers and cloud computing platforms.

Keywords: Bargaining Theory, Kalai-Smorodinsky Solution, Leontief

1. Introduction

The Kalai-Smorodinsky solution has been a subject of much study, since its suggestion by Raiffa [18, 19] and subsequent characterization in terms of monotonicity by Kalai and Smorodinsky [13]. Following this, Moulin [15] showed the implementability of the Kalai-Smorodinsky solution in a bargaining game while Thomson [23] provided a characterization based on population monotonicity. Further implementability characterizations include those by Anbarci [2] and Bossert and Tan [7]. In addition Anant and Mukherji [1] and Conley and Wilkie [9] discussed the Kalai-Smorodinsky solution on non-convex sets.

In this paper we present a new characterizations of the Kalai-Smorodinsky solution. This approach is different in spirit from the previous solutions, including those based on implementability (and related characterizations for other bargaining solutions). Ours is based on truthful allocation mechanisms on Leontief domains. This connects the Kalai-Smorodinsky solution to work on such mechanisms [4] and in addition considers the viewpoint suggested by Binmore [5, 6] and explored in Roemer [20], that one should look beyond the feasible region to the underlying processes that generate the bargaining problems. Related approaches have previously considered connections between allocation mechanisms on underlying economies [6, 20, 22]; however, these typically study axioms that parallel the known characterizations, while our approach uses orthogonal axioms.

Our analysis is motivated by practical concerns arising in modern computing systems, such as data-centers and cloud computing [3, 10]. In that setting, computational tasks have resource requirements which are accurately modeled by Leontief Utility functions and the Kalai-Smorodinsky-solution arises naturally from a mixture of strategic and algorithmic concerns which we discuss briefly in Section 4. (See [10, 11] for additional details.)

Our first result shows that if a truthful, symmetric, efficient and strategyproof allocation mechanism for a two player economy under Leontief preferences induces a bargaining solution then that solution must be the Kalai-Smorodinsky solution. This is somewhat surprising as the set of truthful, symmetric and efficient allocation mechanisms for an economy under Leontief preferences is quite large, as shown by Nicolo [16] and Ji and Xue [14].

In some sense, our analysis connects allocation mechanisms with welfarism, the assumption that bargaining solutions only depend on the geometry of the feasible region and not the underlying economy. Welfarism has been criticized in several contexts [21, 20], but is actually appears to be quite mild in our setting, as we only require that the mechanism be independent of goods which are overabundant – there is no feasible allocation in which they bind. Nonetheless, it is sufficient for our analysis.

We then extend this result to problems with more than two players. As is typical in the study of the Kalai-Smorodinsky solution, extensions to many players are surprisingly complex. We first consider the class of weighted DRF mechanisms which are of great practical interest and are group strategyproof. We show that they satisfy and are characterized by a new proportional consistency axiom, Ratio Consistency, combined with strategyproofness. We show that the multi-player Raiffa solution is the unique weighted DRF mechanism that generates a bargaining solution. Finally, we present a result that complements Imai's characterization of the multi-player lexicographic-Kalai-Smorodinsky solution. We show that strategyproofness, combined with other basic axioms, implies a version of the Individually Rational Individual Monotonicity which when combined with the Independence of Irrelevant Alternatives other than Ideal Point axiom characterizes the lexicographic-Kalai-Smorodinsky solution for 3-players and is conjectured to characterize it for an arbitrary number of players.

These results shed new light and provide new insights into the study of mechanisms on Leontief economies and bargaining theory as well as having practical applications in the design of modern data-centers and cloud computing platforms [10, 11].

2. Allocation and Bargaining

In the following for any integer k, let $\Re^k_+ = \{x \in \Re^k \mid \forall i, x_i \geq 0\}$ while $\Re^k_{++} = \{x \in \Re^k \mid \forall i, x_i > 0\}$.

2.1. Allocation and Leontief Economies

We first define the Leontief economy. Let N be a finite set of players, G a finite set of goods and $s \in \Re^G_{++}$ the supply of each good. A feasible allocation is a vector $x \in \Re^{NG}_+$ such that for all $j \in G$,

$$\sum_{i \in N} x_{ij} \leq s_j$$

and let $X(N, G, s)$ be the set (simplex) of all feasible allocations. We consider Leontief preferences, defined by λ where $\lambda_i \in \Re^G_{++}$ for all $i \in N$ and the utility for player $i \in N$ is given by

$$u_i(x_i, \lambda_i) = \min_{j \in G} x_{ij}/\lambda_{ij}.$$

Note that one can also consider the case with some $\lambda_i = 0$. For simplicity, we will focus on the case where $\lambda_{ij} > 0$ except where specifically stated.

Unlike many allocative analyses we are not going to explicitly assume affine invariance of utility, but instead, as is common in bargaining theory, make no prior assumption about the scale and zero of a utility function. An economy is a tuple

$$E =< N, G, s, \lambda >$$

and let $\mathcal{E}$ be the set of all such economies.

We study the class of mechanisms, $M \in \mathcal{M}$, $M : \mathcal{E} \to X$, where $M(E)_i \in \Re^G_+$ is the vector of goods allocated to agent $i \in N$. We will consider the following axioms for our mechanisms. First we require symmetry with respect to both players and goods.

Axion 1 (Symmetry). *A mechanism is Player Symmetric if for all permutations π of N, $i \in N$, and economies $E =< N, G, s, \lambda > \in \mathcal{E}$*

$$M(\pi(N), G, s, \pi^N(\lambda) >)_i = (M(< N, G, s, \lambda >))_{\pi(i)},$$

where $\pi^N(\lambda)$ is used to show that only the $i \in N$ indices are permuted in λ, not the $j \in G$ indices.
A mechanism is Good Symmetric if for all permutations π of G, $j \in G$, and economies $E =< N, G, s, \lambda > \in \mathcal{E}$

$$M(N, \pi(G), \pi(s), \pi^G(\lambda >))_i = (M(< N, G, s, \lambda >))_{\pi(i)},$$

where $\pi^G(\lambda)$ is used to indicate that only the $j \in G$ indices are permuted in λ, not the $i \in N$ indices.
A mechanism is Symmetric if it is both Player Symmetric and Good Symmetric.

Second, we require that the units in which goods are measured do not affect the outcome.[1] Given some $\rho \in \Re^G_{++}$ and any $i \in N$, $j \in G$ and $x_i \in \Re^G_+$ define the rescaling operator $S_\rho(x_i)_j = \rho_j x_{ij}$.

[1]Note that this is an important distinction between our analysis and that in [14], as their mechanisms depend strongly on the units.

Axion 2 (Scale Invariance of Goods). *A mechanism is scale invariant if for all $\rho \in \Re^G_{++}$, $i \in N$, $j \in G$ and economies $E =< N, G, s, \lambda > \in \mathcal{E}$*

$$M(N, G, S_\rho(s), \lambda >)_{ij} = \rho_{ij} M(< N, G, s, \lambda >)_{ij}.$$

Note that if a mechanism is scale invariant, then we can always assume that for all $j \in G$, $s_i = 1$, something which we will do throughout the remainder of this paper in order to simplify notation.

One might also consider scale invariance of the utility functions, which would imply that given some λ_i scalar multiples of it, $\alpha\lambda_i$ should not affect the allocation. However, this is not universally assumed in bargaining theory so we will not define it explicitly. Interestingly, we will see that it arises naturally from strategic considerations.

Next, we define

$$P(a) = \{x \in X_m \mid \nexists x' \neq x \in X, \ s.t.\ \forall i \in N,\ U_i(x') \geq U_i(x) \ \ \&\ \exists i \in N,\ U_i(x') > U_i(x)\}$$

to be the strict Pareto frontier and require the following.

Axion 3 (Efficiency). *A mechanism is Efficient if for all $E \in E$, $M(E) \in P(E)$.*

Finally, we require strategyproofness.

Axion 4 (Strategyproofness). *A mechanism is Strategyproof if for all $i \in N$, $E =< N, G, s, \lambda > \in \mathcal{E}$, and for any $i \in N$ and $\lambda'_i \in \Re^G_{++}$,*

$$u_i(M(E); \lambda_i) \geq u_i(M(< N, G, s, (\lambda'_i, \lambda_{-i}) >); \lambda_i).$$

Note that even in the case of 2 players, there are many strategyproof mechanisms [16, 14], even in the when there are only 2 goods. Somewhat surprisingly many of these mechanisms are also group strategyproof as we will show in the next section.

Axion 5 (Group Strategyproofness). *A mechanism is Strategyproof if for all $i \in N$, $E =< N, G, s, \lambda > \in \mathcal{E}$, and for any $S \subset N$ and $\lambda'_S \in \Re^{SG}_{++}$ for which*

$$u_i(M(E); \lambda_i) < u_i(M(< N, G, s, (\lambda'_S, \lambda_{-S}) >); \lambda_i)$$

for some for some $i \in S$, there exists some $\hat{i} \in S$ such that

$$u_{\hat{i}}(M(E); \lambda_{\hat{i}}) > u_i(M(E'); \lambda_i),$$

when $E' =< N, G, s, (\lambda'_S, \lambda_{-S}) >$.

In addition, note that for Leontief preferences many allocations are wasteful in the sense that one can often reduce x_{ij} by a small amount without reducing player i's utility. This arises whenever x_i is not a multiple of λ_i. Given a wasteful mechanism M one can remove the waste without reducing any player's utility. For any mechanism let $R(M)$ be the unique non-wasteful mechanism that implements the same utility profile as M for all $E \in \mathcal{E}$. Note that [16] considers wasteful mechanism in which all goods are allocated while [14] considers only non-wasteful mechanisms, allowing for stronger results in the latter. It is easy to see that for any wasteful mechanism M that satisfies any of the above axioms its non-wasteful counterpart $R(M)$ satisfies the same axioms. In particular, if M is strategyproof (or group strategyproof), then so is $R(M)$. Thus, we will consider only non-wasteful mechanisms in the remainder of this paper.

2.2. Bargaining Problems

We can also view this problem as a bargaining problem over the feasible utility profiles,

$$\mathbf{U}(E) = \{u(x;\lambda) \mid x \in X(E)\}$$

with the disagreement point set to 0. Note that many mechanisms do not induce a bargaining solution since there are be many different economies which correspond to the same set of feasible utilities. Our interest throughout most of this paper is the connection between mechanisms and bargaining, thus we will consider mechanisms that do induce a bargaining solution on $\mathbf{U}(a)$. (Note that here and in the following we will assume that the disagreement point is always 0.)

Axion 6 (IBS). *The mechanism Induces a Bargaining Solution: For all $E, E' \in \mathcal{E}$ such that $\mathbf{U}(E) = \mathbf{U}(E')$, $U(M(E);\lambda) = U(M(E');\lambda')$.*

Note that in the IBS axiom E and E' *do not* necessarily have the same number of goods; however, the differences must always correspond to goods which are never strictly binding in any non-wasteful allocation.

Given our assumptions, the set of feasible (utility) regions $\mathcal{U}$ is the set of non-negative strictly comprehensive convex polytopes. In addition, these polytopes have the property that any weakly Pareto efficient point must also be strongly Pareto efficient.

Recall that the ideal points of a bargaining problem $I(\mathbf{U}(E))$ are given by

$$I_i(\mathbf{U}(E)) = \max\{u_i \mid u \in \mathbf{U}(E)\} = \max\{u_i(x_i, \lambda_i) \mid x \in X(E)\}$$

and in our case this is given simply by $I_i(\mathbf{U}(E)) = \max_j \lambda_{ij}$ since we will only consider the case where all $s_i = 1$. The Raiffa solution is then maximum value of $\kappa I(\mathbf{U}(E))$ over $\mathbf{U}(E)$, and we define $\kappa(E)$ to be the value of κ that attains this maximum. The corresponding non-wasteful allocation is simply given by $x_i = \kappa(E)\lambda_i$ for all $i \in N$. Alternatively one can compute $\kappa(E)$ by maximizing

$$\max_\kappa \sum_i \kappa\lambda_i / I_i(E)$$

over $X(E)$. In the case that $|N| = 2$ this is also the 2-player Kalai-Smorodinsky solution. We will denote the allocation mechanism the Kalai-Smorodinsky mechanism and denote it by M^κ.

3. Strategyproofness and the Kalai-Smorodinsky solution

Our first result shows that one can use strategyproofness to characterize the Kalai-Smorodinsky solution in the standard case with two players.

Theorem 1. *A mechanism $M \in \mathcal{M}$ on $\mathcal{E}$ satisfies, Symmetry, Scale Invariance of Goods, Efficiency, Strategyproofness and Induces a Bargaining Solution if and only if it the Kalai-Smorodinsky mechanism M^κ and induces the Kalai-Smorodinsky solution κ.*

Proof: It is easy to see from the definition that M^κ satisfies, Symmetry, Scale Invariance of Goods, Efficiency and Induces a Bargaining Solution. It is also strategyproof as it is a special case of Theorem 3 in the next section.

To show that it is unique, consider the Economy $E \in E$ where $N = \{1, 2\}$, $G = \{1, 2, \ldots, m\}$, $s = e$ and $\lambda \in \Re^{NG}_{++}$. Let M be the mechanism and recall that

$$M^\kappa(E) = (\kappa(E)\lambda_1, \kappa(E)\lambda_2).$$

Now consider the augmented problem, E' where E' agrees with E everywhere except we add 2 additional goods $m+1, m+2$ to E'. Let $\lambda'_{\cdot,m+1} = (\epsilon, 1)$ and $\lambda'_{\cdot m+2} = (\epsilon, \gamma)$ where

$$\epsilon = \min_i \min_j \lambda_{ij}/2$$

and

$$\gamma = \kappa(E)(2\kappa(E) - 1)/(1 - \kappa(E)).$$

Since M induces a bargaining solution,

$$U(M(E); \lambda) = U(M(E); \lambda'),$$

so we only need to consider E'.

Now consider the deviation by player 1 where instead of reporting her true values in $\lambda_{1,m+1}$ and $\lambda'_{1,m+2}$ she reports $\hat{\lambda}'_{1,m+1} = \gamma$ and $\hat{\lambda}'_{1,m+2} = 1$. It is easy to check that the feasible region under this deviation is simply the convex hull of $(1, 0)$, $(0, 1)$ and $\kappa(E')(1, 1)$ which is also generated by the the 2 good economy E'' with

$$\lambda_1 = (1, \kappa(E'))$$

and

$$\lambda_2 = (\kappa(E'), 1).$$

Then, by Symmetry and Pareto efficiency, $M(E'')$ must be Symmetric and thus give the same utility as $U(M^{\kappa}(E); \lambda)$, which implies that the solution is the Kalai-Smorodinsky solution.

Thus, in the original economy E player 1 must attain at least as much utility as in the Kalai-Smorodinsky solution. However, we can repeat this argument for player 2 to show that he too must attain at least the utility level of the Kalai-Smorodinsky solution. Since the Kalai-Smorodinsky solution is Pareto efficient this implies that the outcome for the original economy must be the Kalai-Smorodinsky solution. □

Note that the analysis is unchanged if we allow for the extended domain in which we allow some $\lambda_{ij} = 0$, i.e. there are goods that never contribute to a player's utility.

4. Weighted DRF Mechanisms

The Dominant Resource Fairness (DRF) mechanism was analyzed in [10] in the context of allocations for data-centers and cloud computing [11]. It is a dynamic mechanism that allocates resource requests in real time in a stochastic dynamic system. On average, in steady state, it computes the egalitarian equivalent allocation originally proposed by Pazner and Schmeidler [17]. The DRF mechanism is known as a "water-filling" algorithm, a class of algorithms which are common in many settings [24]. Below we define a class of such mechanisms which are weighted versions of the DRF mechanism and retain many of the strong properties of that mechanism.

Given any symmetric weight function $w(\lambda_i) \in \Re_+$ define the weighted-DRF allocation to be defined by the value $\kappa^w(E)$ that maximizes

$$\kappa \sum_{i \in N} w(\lambda_i)\lambda_i$$

over $X(E)$. It then allocates goods according to $M^w(E)_i = \kappa(E)w(\lambda_i)\lambda_i$. If we set $w(\lambda_i) = (\max_j \lambda_{ij})^{-1}$, then $M^w = M^\kappa$, the Kalai-Smorodinsky mechanism.

Weighted DRF mechanisms are not consistent in the standard sense[2] but satisfy a strong fairness property which is related to consistency. The basic idea is to compare utility profiles according to a fixed "exchange rate".

Axion 7 (Ratio Consistency). *A bargaining solution f is Ratio Consistent if for any two players, $i, \hat{i}$ and any two bargaining sets $\mathbf{U}$ and $\mathbf{U}'$ with identical projections onto $\Re^{i\hat{i}}$ the ratios are constant,*

$$\frac{f(\mathbf{U})_i}{f(\mathbf{U})_{\hat{i}}} = \frac{f(\mathbf{U}')_i}{f(\mathbf{U}')_{\hat{i}}}.$$

A mechanism M is Ratio Consistent if for any two players for any economy $E \in \mathcal{E}$ and $i, \hat{i} \in N$, there exists some $\rho > 0$ such that for any economy $E' \in \mathcal{E}$ with $N \subseteq N'$, and both $\lambda_i = \lambda'_i$ and $\lambda_{\hat{i}} = \lambda'_{\hat{i}}$, there exists r_i and $r_{\hat{i}}$ satisfying $M(E)_i = r_i M(E')_i$ and $M(E)_{\hat{i}} = r_{\hat{i}} M(E')_{\hat{i}}$ where $\rho = r_i / r_{\hat{i}}$.

Note that these two variations on ratio consistency are consistent with each other.

Lemma 1. *If a mechanism $M \in \mathcal{M}$ is ratio consistent and induces a bargaining solution, then the induced bargaining solution is ratio consistent.*

It is easy to see that weighted DRF mechanisms are ratio consistent, but it is also true that they can be characterized by ratio consistency and the other axioms.

[2] Compare this to [14] who study consistent, but non-scale invariant mechanisms for this economy.

Theorem 2. *A mechanism $M \in \mathcal{M}$ on $\mathcal{E}$ satisfies, Symmetry, Scale Invariance of Goods, Efficiency, and Ratio Consistency if and only if it is a weighted DRF mechanism.*

Proof: First note that by construction weighted DRF mechanisms satisfy Symmetry, Scale Invariance and Efficiency. To see that they satisfy Ratio Consistency, note that the weighted DRF allocations lie on the line defined by

$$\kappa \sum_{i \in N} w(\lambda_i)\lambda_i$$

for any economy and thus the ratio between any two players with λ_i and $\lambda_{\hat{i}}$ is simply $w(\lambda_i)/w(\lambda_{\hat{i}})$.

For the converse, given some mechanism satisfying the given axioms, and given any set G and some $\lambda_1 \in \Re^G_{++}$ consider the economy $E =< N, G, s, \lambda >\in \mathcal{E}$ where $N = 1, 2$, $s = e$, and $\lambda_2 = e$. Set $w(e) = 1$ and

$$w(\lambda_1) = \frac{u_1(M^w(E)_1, \lambda_1)}{u_{\hat{i}}(M^w(E)_2, \lambda_2)}.$$

To check that $M^w = M$ next extend to a 3 player economy with any $\lambda_1, \lambda_2 \in \Re^G_{++}$ and $\lambda_3 = e$. This fixes the ratio between players with preferences defined by $\lambda_1, \lambda_2 \in \Re^G_{++}$. Lastly, given an economy with an arbitrary number of players, note that the ratio between all pairs is fixed and this, combined with Efficiency uniquely defines the mechanism M^w. □

If we add some requirements to the weight functions, then these mechanisms also have strong strategic properties. A continuous unordered weighted DRF mechanism is a weighted DRF mechanism where $w(\lambda_i)$ is continuous and unordered in the sense that for any $\lambda_i, \lambda'_i \in \Re^G_{++}$ either

$$m(\lambda_i)\lambda_i = m(\lambda'_i)\lambda'_i$$

(which occurs when $\lambda_i = \alpha\lambda'_i$ for some $\alpha > 0$) or $m(\lambda_i)\lambda_i$ and $m(\lambda'_i)\lambda'_i$ are not ordered, i.e. neither

$$m(\lambda_i)\lambda_i \geq m(\lambda'_i)\lambda'_i$$

nor

$$m(\lambda_i)\lambda_i \leq m(\lambda'_i)\lambda'_i$$

when

$$m(\lambda_i)\lambda_i \neq m(\lambda'_i)\lambda'_i.$$

Note that unorderedness implies that $w(\cdot)$ is homogeneous with degree -1, i.e. for any $\alpha > 0$,

$$w(\alpha\lambda_i) = \alpha^{-1}w(\lambda_i).$$

Theorem 3. *A continuous unordered weighted DRF mechanism on $E \in \mathcal{E}$ is Group Strategyproof.*

Proof: Consider a subset $S \subseteq N$ and two economies, $E, E' \in \mathcal{E}$ where the only difference between E and E' is that some of the λ_i's for $i \in S$ have changed.

First consider a deviation in which $\kappa(E)$ is not decreased. F or any $j \in G$ this implies that $\sum_{i \notin S} x_{ij}$ does not decrease so $\sum_{i \in S} x_{ij}$ can not increase, which implies that if the utility of any player in S increases then some other player in S must have her utility decrease.

Next consider a deviation in which $\kappa(E)$ strictly decreases. Now, agent $i \in S$ receives $M_i^w(E') = \kappa(E')w(\lambda_i')\lambda_i'$ compared to $M_i^w(E) = \kappa(E)w(\lambda_i)\lambda_i$. If this were a profitable deviation then it must be the case that

$$\kappa(E')w(\lambda_i')\lambda_i' \geq \kappa(E)w(\lambda_i)\lambda_i,$$

but since $\kappa(E') < \kappa(E)$ this implies that

$$w(\lambda_i')\lambda_i' > w(\lambda_i)\lambda_i,$$

which contradicts the unorderedness of $w(\cdot)$. Thus, the mechanism is group-strategyproof. □

If we allow for transferable utility, then the DRF mechanism (or its weighted counterparts) is not strategyproof in a stronger version of group strategyproofness where we allow the transfer of goods among the deviating group. Consider an example with 3 players and 4 goods where

$$\lambda_1 = (1, 1/10, 1/10, 1/2),$$

$$\lambda_2 = (1/10, 1, 1/10, 1/2)$$

and

$$\lambda_3 = (1/10, 1/10, 1, 1/2).$$

If all agents are truthful then the outcome under DRF will have $\kappa(E) = 2/3$ and be

$$x_1 = (2/3, 1/15, 1/15, 1/3),$$

$$x_2 = (1/15, 2/3, 1/15, 1/3),$$

and

$$x_3 = (1/15, 1/15, 2/3, 1/3).$$

However, if players 2 and 3 collude and report $\lambda_2' = (1/5, 1, 2/3, 3/4)$ and

$$\lambda_3' = (1/5, 2/3, 1, 3/4)$$

then the outcome under DRF will have $\kappa(E) = 1/2$ and now

$$\hat{x}_2 = (1/10, 1/2, 1/3, 3/8),$$

and

$$\hat{x}_3 = (1/10, 1/3, 1/2, 3/8).$$

Now

$$\hat{x}_2 + \hat{x}_3 = (3/15, 5/6, 5/6, 3/4)$$

whereas before the deviation

$$x_2 + x_3 = (2/15, 11/15, 11/15, 2/3).$$

Thus, the total goods allocated to the deviators has strictly increased, making it a profitable deviation, which strictly increases both deviators' utility after an appropriate division of the goods.

We can now combine these results to characterize the n-player Raiffa Solution.

Theorem 4. *A mechanism $M \in M$ on $E \in \mathcal{E}$ satisfies, Symmetry, Scale Invariance, Efficiency, Ratio Consistency, Strategyproofness and Induces a Bargaining Solution if and only if it is the weighted DRF mechanism (which induces the Raiffa Solution).*

Proof: First, combining the Theorem 2 and Theorem 3 shows that the DRF mechanism satisfies all the assumptions of the theorem and that the DRF mechanism induces the Raiffa Solution. For the converse, Theorem 2 shows that the mechanism must be a weighted DRF mechanism.

In order to be strategyproof the weight function must be continuous and homogeneous of degree -1 by standard arguments: If the weighting has a discontinuity at some λ then there exists a infinitesimal deviation that has a finite effect on the weight function which either increases or decreases utility and thus in one direction, for a small enough deviation, will be profitable, while if the weighting is not homogeneous with degree -1 then there exists some λ and some $\alpha > 0$ such that the deviation from λ_i to $\alpha\lambda_i$ is profitable, since in both cases the allocation will be a different multiple of λ_i.

Now assume that w does not generate the DRF mechanism and suppose that there are $\lambda_1, \lambda'_1 \in \Re^G_{++}$ normalize (by homogeneity) such that $\max_{j\in G} \lambda_{1j} = 1$ and $\max_{j\in G} \lambda'_{1j} = 1$ which disagree on only one component, $\hat{j} \in G$ but $w(\lambda_1) \neq w(\lambda'_1)$. Consider a pair of two player economies E with the same set of goods and λ_1 in the first, λ'_1 in the second and a second player with λ_2 in both where

$$\lambda_{2\hat{j}} < 1 - min[\lambda_{1\hat{j}}, \lambda'_{1\hat{j}}].$$

Since $w(\lambda_1) \neq w(\lambda'_1)$ the allocation will be different E and E'. However, the constraint for good $\hat{j}$ does not affect the feasible region since $\lambda_{1\hat{j}} + \lambda_{2\hat{j}} < 1$ and $\lambda'_{1\hat{j}} + \lambda_{2\hat{j}} < 1$, which shows that this mechanism would not induce a bargaining solution since it does not depend only on the feasible region, thus providing a contradiction and completing the proof. □

5. Strategyproofness and Lexicographic-Kalai-Smorodinsky solution

Perhaps the strongest multi-player generalization of the Kalai-Smorodinsky solution is due to Imai [12] who describes the lexicographic-Kalai-Smorodinsky solution (lKS) which is defined on the extended domain where $\lambda_i \in \Re^G_+$ on which a player's utility only depends on a subset of the goods, i.e. goods for which $\lambda_{ij} = 0$ are just ignored in player i's utility function.[3] In this extended domain, which we denote by $\mathcal{E}^0$, the Raiffa Solution (and two player Kalai-Smorodinsky solution) is only weakly Pareto Efficient.

The lexicographic order of two vectors, $v, v' \in \Re^N$ is defined by first permuting each vector so that the elements are in nondecreasing order. Let $P(\cdot)$ be the ordering function which does this. Given two ordered vectors $P(v)$ and $P(v')$ let j be the first element on which they disagree, $P(v)_j \neq P(v')_j$ then the lexicographic order of the vectors v, v' is the same as the regular order of $P(v)_j, P(v')_j$. For example if $v = (3, 1, 1, 2)$ and $v' = (1, 2, 4, 3)$ then

[3] Alternatively, we can define $a/0 = \infty > b$ for all $a \geq 0$ and $b \in \Re$.

$P(v) = (1,1,2,3)$ and $P(v') = (1,2,3,4)$ so $j = 2$ and v is smaller than v' in the lexicographic ordering. Recall that under the lexicographic ordering any 2 vectors are comparable and two vectors are equal in the ordering only if they are permutations of each other.

Using this, the lKS solution lexicographically maximizes the vector ξ over the set

$$\{(\xi_1 I_1(E), \xi_2 I_2(E), \ldots, \xi_n I_n(E)) \in \mathbf{U}\}$$

when the players are $N = \{1, 2, \ldots, n\}$ for $E \in \mathcal{E}^0$. The related lKS mechanism M^ξ, maximizes ξ over

$$\sum_{i \in N} \xi_i \lambda_i / I_i(E) \in \mathcal{X}(E)$$

for $E \in \mathcal{E}^0$.

Imai [12] characterizes the lKS using the following axioms which are analogous axioms that we considered for economies, but stated in the terms of the bargaining set, where $\mathcal{U}$ is the set of all bounded and convex subsets of $\Re^N$ containing the origin. (Note that we are assuming that the origin is the disagreement point for simplicity.)

Axion 8. *A bargaining solution f is Efficient if $f(\mathbf{U})$ is Pareto Efficient for all $\mathbf{U} \in \mathcal{U}$.*

Axion 9. *A bargaining solution f is Symmetric if $\pi(f(\mathbf{U})) = f(\pi(\mathbf{U}))$ for all permutations π and all $\mathbf{U} \in \mathcal{U}$.*

Axion 10. *A bargaining solution f is Scale Invariant if $S_\rho(f(\mathbf{U})) = f(S_\rho(\mathbf{U}))$ for all permutations $\rho \in \Re^N_{++}$ and all $\mathbf{U} \in \mathcal{U}$.*

He then adds a variant on Nash's Independence of Irrelevant Alternatives axiom.

Axion 11 (IIIA). *A bargaining solution f satisfies Independence of Irrelevant Alternatives other than Ideal Point if for all $\mathbf{U}, \mathbf{U}' \in \mathcal{U}$ with $\mathbf{U} \subseteq \mathbf{U}'$ and $I(\mathbf{U}) = I(\mathbf{U}')$ if $f(\mathbf{U}') \in \mathbf{U}$ then $f(\mathbf{U}) = f(\mathbf{U}')$.*

Lastly, he considers a generalization of Kalai-Smorodinsky's monotonicity axiom. Given a set $\mathbf{U} \in \mathcal{U}$ for any $i \in N$ define ${}_i\mathbf{U}$ to be the projection of $\mathbf{U}$ onto $\Re^{N \setminus i}$.

Axion 12 (IM). *A bargaining solution f satisfies Individual Monotonicity if for all players $i \in N$ and all $\mathbf{U}, \mathbf{U}' \in \mathcal{U}$ with $\mathbf{U} \subseteq \mathbf{U}'$ and ${}_i\mathbf{U} =_i \mathbf{U}'$ then $f_i(\mathbf{U}) \leq f_i(\mathbf{U}')$.*

He then proves his main result.

Theorem 5 (Imai, 1983). *The lKS solution is the unique bargaining solution satisfying Efficiency, Symmetry, Scale Invariance, IIIA and IM.*

As pointed out by Imai, this Individual Monotonicity axiom used in the characterization is flawed since the projection depends on the full bargaining set and not just the individually rational part of it. In particular, the proof uses IIIA to consider on bargaining sets which are fully comprehensive. This is somewhat unnatural and leads to problems when we restrict the class of bargaining sets in natural ways. For example as Imai states [12] "for the class of problems with individually rational agreement sets, our proof of the uniqueness of the solution does not apply." Essentially, in Imai's proof of the theorem, the projection requirement is only needed to rule out a few "degenerate" reductions $\mathbf{U} \subseteq \mathbf{U}'$ of the type where we take the intersection of $\mathbf{U}'$ with a hyperplane that has a normal vector with zero components.

To repair this, in our setting, one can consider only the individually rational part of the bargaining set, $\mathbf{U}_d$ and then modify the axiom as follows.

Axion 13 (IRIM). *A bargaining solution f satisfies Individually Rational Individual Monotonicity if for all players $i \in N$ and all $\mathbf{U}, \mathbf{U}' \in \mathcal{U}$ with $\mathbf{U} \subseteq \mathbf{U}'$ and ${}_i\mathbf{U}_d =_i \mathbf{U}'_d$ then $f_i(\mathbf{U}) \leq f_i(\mathbf{U}')$.*

This axiom is much more natural, but much more difficult to analyze. The characterization using this axiom has only been accomplished for $|N| \leq 3$ by Chang and Liang [8].

Theorem 6 (Chang and Liang, 1998). *For $|N| \leq 3$ the lKS solution is the unique bargaining solution satisfying Efficiency, Symmetry, Scale Invariance, IIIA and IRIM.*

In general the multi-player generalization is still a conjecture.

Conjecture 1 (Imai-Chang-Liang). *The lKS solution is the unique bargaining solution satisfying Efficiency, Symmetry, Scale Invariance, IIIA and IRIM.*

We now show that the proof of this conjecture would provide an interesting strategic characterization of the lKS mechanism.

Theorem 7. *For $|N| \leq 3$ a mechanism $M \in M$ on $E \in \mathcal{E}^0$ satisfies, Symmetry, Efficiency, IIIA, Strategyproofness and Induces a Bargaining Solution if and only if it induces the lKS Solution. If the Imai-Chang-Liang conjecture is true then this characterization holds for all N.*

Proof: To show the equivalence of this theorem to the theorem of Chang and Liang, and the conjecture, we need to show that these axioms imply Scale Invariance and IRIM. The first is simple as our analysis in the proof of Theorem 1 shows that strategyproofness implies Scale Invariance. We need to show that it also implies IRIM.

In fact, as a close examination of Imai's proof shows, we only need a weaker version of the IRIM axiom. The axiom, as written, considers any two subsets $\mathbf{U}, \mathbf{U}' \in \mathcal{U}$ which satisfy $\mathbf{U} \subseteq \mathbf{U}'$ and ${}_i\mathbf{U}_d =_i \mathbf{U}'_d$; however, all that is required for Imai's proof are sets of the form $\mathbf{U}, \mathbf{U}' \in \mathcal{U}$ which satisfy $\mathbf{U} = \mathbf{U}' \cap H(p)$ where $H(p)$ is a half-space, i.e. $H(p) = \{x \in \Re^N \mid p \cdot x \leq 1\}$.

Consider an economy $E \in \mathcal{E}^0$ which generates $\mathbf{U}(E)$ and an agent $\hat{i} \in N$. Suppose that p generates a half space and let $\mathbf{U} = \mathbf{U}' \cap H(p)$ such that ${}_{\hat{i}}\mathbf{U}_d =_{\hat{i}} \mathbf{U}'_d$. Now there exists some $\epsilon > 0$ which is sufficiently small such that if we define p' by $p'_{-\hat{i}} = p_{-\hat{i}}$ and $p_{\hat{i}} = \epsilon$ then $\mathbf{U}' \cap H(p') = \mathbf{U}'$. So by the IBS assumption we can add a new good j and set $\lambda_{ij} = p_i$ for all $i \in n$ to get a new economy $E' \in \mathcal{E}$ such that $M(E) = M(E')$. Now, consider the deviation by $\hat{i}$ where she pretends that $\lambda'_{\hat{i}j} = p_{\hat{i}}$ to get the economy $\hat{E} \in \mathcal{E}$. Since this deviation can not be profitable by strategyproofness this must imply that

$$u(M(\hat{E}_{\hat{i}}), \lambda'_{\hat{i}}) \leq u(M(E_{\hat{i}}), \lambda_{\hat{i}})$$

which also implies that

$$u(M(\hat{E}_{\hat{i}}), \lambda_{\hat{i}}) \leq u(M(E_{\hat{i}}), \lambda_{\hat{i}})$$

since $\lambda_{\hat{i}}$ and $\lambda'_{\hat{i}}$ only disagree on a single component. Thus the induces bargaining solution must satisfy IRIM completing the proof. □

Thus, there is a strong connection between strategyproofness and IRIM.

6. Conclusions

We have shown several interesting connections between bargaining solutions and strategyproofness on the underlying economies. Our work suggests a variety of directions for further study, such as the extension to different classes of economies and variations on the underlying axiom. In addition, it suggests that one might be able to use insights from mechanism design to understand bargaining solutions. For example, it might be possible to prove the difficult Imai-Chang-Liang conjecture by a direct analysis on an underlying economy. Our work also provides a tool for the development of robust and non-manipulable algorithms for a variety of practical problems arising in data centers and cloud computing.

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