Isoperimetric Functions of Amalgamations of Nilpotent Groups

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Abstract

We consider amalgamations of finitely generated nilpotent groups of class c. We show that doubles satisfy a polynomial isoperimetric inequality of degree $2c^2$. Generalising doubles we introduce non-twisted amalgamations and we show that they satisfy a polynomial isoperimetric inequality as well. We give a sufficient condition for amalgamations along abelian subgroups to be non-twisted and thereby to satisfy a polynomial isoperimetric inequality. We conclude by giving an example of a twisted amalgamation along an abelian subgroup having an exponential isoperimetric function.

1 Introduction

1.1 Isoperimetric Functions

The isoperimetric function of a finitely presented group G limits the number of defining relators needed to show that a word represents the identity in G. Hence the isoperimetric function is a measure for the complexity of the word problem. Suppose $G = F/R$ where F is a free group freely generated by the finite set $\mathcal{F}$ and R is the normal closure of a finite set of relators $\mathcal{R} \subset F$. Thus $P = \langle \mathcal{F} \mid \mathcal{R} \rangle$ is a finite presentation of G. For short we identify words $w \in F$ with their residue classes $wR \in G$. A word w is equal to 1 in G if and only if w is freely equal to a word of the form

$$\prod_{i=1}^{m} u_i^{-1} r_i^{\epsilon_i} u_i \quad \text{with } u_i \in F,\ r_i \in \mathcal{R} \text{ and } \epsilon_i = \pm 1.$$

Let $\Delta_P : R \to N$ be the so-called *area function* defined by

$$\Delta_P(w) = \min\{m \in N \mid w = \prod_{i=1}^{m} u_i^{-1} r_i^{\epsilon_i} u_i \text{ for } u_i \in F,\ r_i \in \mathcal{R}, \epsilon_i = \pm 1\}$$

for a word $w \in R$. We denote by $|w|$ the length of w. Associated with Δ_P is the *isoperimetric function* Φ_P *of the finite presentation* P defined by

$$\Phi_P(n) = \max\{\Delta_P(w) \mid w \in R \text{ and } |w| \leq n\}.$$

A partial ordering $\preceq$ on functions on the natural numbers is used to compare isoperimetric functions. For $f, g : N \to N$ let $f \preceq g$ if and only if there exists a constant K such that $f(n) \leq Kg(Kn) + Kn$ for all $n \in N$. Hence we get an equivalence relation $\cong$ where $f \cong g$ if and only if $f \preceq g$ and $g \preceq f$. If P and Q are different finite presentations of the same group then $\Phi_P \cong \Phi_Q$, cf. [Alo90]. Any $N \to N$ function equivalent to Φ_P is called an *isoperimetric function of* G, denoted by Φ_G. We say that G *satisfies a polynomial isoperimetric inequality of degree* k if Φ_G is bounded above by a polynomial of degree k.

We note that all finitely generated nilpotent groups are finitely presented. For a finitely generated free nilpotent group G of class c we have $n^{c+1} \preceq \Phi_G$ by [BMS93, Ger93]. Pittet shows in [Pit95], based on [Gro93, $5.A_2'$], that $\Phi_G \preceq n^{c+1}$. Hence the isoperimetric function of a free nilpotent group of class c is equivalent to n^{c+1}. For an arbitrary finitely generated nilpotent group G of class c we have $\Phi_G \preceq n^{2c}$, cf. [Hid97].

1.2 Isoperimetric Functions of Amalgamations

Let G_i for $i = 1, 2$ be a group finitely presented by $P_i = \langle \mathcal{F}_i \mid \mathcal{R}_i \rangle$, H_i a subgroup of G_i generated by $\mathcal{E} = \mathcal{F}_1 \cap \mathcal{F}_2$ and let the canonical map $w \mapsto w$ with $w \in E$ be an isomorphism between H_1 and H_2. The group given by the finite presentation $P = \langle \mathcal{F}_1 \cup \mathcal{F}_2 \mid \mathcal{R}_1 \cup \mathcal{R}_2 \rangle$ is called the *generalised free product of* G_1 *and* G_2 *amalgamating*

H_1 *and* H_2 or an *amalgamation of* G_1 *and* G_2 *along* H and is denoted by $G_1 *_H G_2$ with $H \cong H_1 \cong H_2$.

Let $G = G_1 *_H G_2$. The following results are contained in [BGSS91]: If G_1, G_2 are abelian then $\Phi_G \preceq n^2$. If G_1, G_2 are automatic then G is asynchronously automatic and thus $\Phi_G \preceq 2^n$. If G_1, G_2 are free and H is cyclic then G is automatic and thus $\Phi_G \preceq n^2$.

If H is finite then $\Phi_G \preceq \bar{\Phi}_{G_1} + \bar{\Phi}_{G_2}$, where $\bar{\Phi}_{G_i}$ is the superadditive closure of Φ_{G_i}, cf. [Bri93].

Let $P = \langle \mathcal{F} \,|\, \mathcal{R} \rangle$ be a finite presentation for a group G and H a subgroup of G generated by $\mathcal{E} \subseteq \mathcal{F}$. Let w be a word in the generators $\mathcal{F}$ representing an element in H and $\rho(w)$ a word of minimal length in the generators $\mathcal{E}$ such that $\rho(w) =_G w$. The function $\delta_{H,G}(n)$ defined by the maximum of $|\rho(w)|$ over all $w \in H$ in the generators $\mathcal{F}$ of length $\leq n$ is called the *distortion of* H *in* G. In analogy to the isoperimetric function, the distortion does not depend up to $\cong$-equivalence on the presentation P, cf. [Far94].

The following results are contained in [Hid97]: Let G_i for $i = 1, 2$ be a finitely presented group satisfying a polynomial isoperimetric inequality, H a finitely generated subgroup of G_i and $G = G_1 *_H G_2$. If H is linearly distorted then $\Phi_G \preceq 2^n$. If H is normal and at most exponentially distorted in G then $\Phi_G \preceq 2^n$. If H is central and at most polynomially distorted in G_i then G satisfies a polynomial isoperimetric inequality. In general, if H is at most polynomially distorted then $\Phi_G \preceq 2^{(2^n)}$. Thus *any amalgamation* $G = G_1 *_H G_2$ *of finitely generated nilpotent groups satisfies a double exponential isoperimetric inequality.* However, if G_i is torsionfree and H cyclic, then G satisfies a polynomial isoperimetric inequality of degree $4c^2$. Our goal is to lower the double exponential upper bound for Φ_G to a polynomial upper bound for the case where G is a double, a non-twisted amalgamation or an amalgamation along a suitable abelian subgroup.

1.3 Rewriting Process

Let G be a finitely presented group, H a finitely presented subgroup of G and w a word of length n equal to 1 in G. Suppose that we already know Φ_H or an upper bound thereof. To compute an upper bound for Φ_G we use the following approach: We rewrite w to a word $\rho(w)$ in the generators of H. We then compute an upper bound $\Phi_\rho(n)$ for the number of relators needed to rewrite w to $\rho(w)$ and an upper bound $\delta_\rho(n)$ for the length of $\rho(w)$. Since $\rho(w) =_G w$ the word $\rho(w)$ is equal to 1 in H as well. The area of $\rho(w)$ is bounded above by $\Phi_H(\delta_\rho(n))$. Therefore the area of w is bounded above by $\Phi_H(\delta_\rho(n))$ plus the number of relators needed to rewrite w to $\rho(w)$. Hence $\Phi_H(\delta_\rho(n)) + \Phi_\rho(n)$ is an upper bound for the isoperimetric function of G.

Let $P = \langle \mathcal{F} \,|\, \mathcal{R} \rangle$ be a finite presentation of the group G, F the free group freely generated by $\mathcal{F}$ and H a finitely generated subgroup of G. We may assume, without loss of generality, that H is generated by a subset $\mathcal{E} \subset \mathcal{F}$. Let E be the subgroup of F generated by $\mathcal{E}$. A *rewriting process ρ from G to H relative to P, $\mathcal{E}$* is a partial map $F \xrightarrow{\rho} E$ defined on all words $w \in H$ such that $\rho(w) =_G w$ and $\rho(1) = 1$. In general ρ is not a homomorphism. Define $\delta_\rho(n)$ by the maximal length of $\rho(w)$ for all $w \in H$ with $|w| \leq n$. We call δ_ρ the *distortion of the rewriting process ρ*. In analogy to Φ_P, we define $\Phi_\rho(n)$ to be the function defined by

$$\max\{\Delta_P(w^{-1}\rho(w)) \,|\, w \in H \text{ and } |w| \leq n\}.$$

We call Φ_ρ the *isoperimetric function of the rewriting process ρ*.

Let ρ_1 be a rewriting process from G_1 to a subgroup $H_1 \subseteq G_1$ relative to some finite presentation and finite set of generators. Suppose ψ is a retraction from G_1 to a finitely presented group G_2. There exists a rewriting process ρ_2 from G_2 to $H_2 = \psi(H_1)$ relative to a given finite presentation for G_2 and finite set of generators for H_2 such that $\delta_{\rho_2} \preceq \delta_{\rho_1}$ and $\Phi_{\rho_2} \preceq \Phi_{\rho_1}$, cf. [Hid97]. Since a group is trivially a retract of itself, upper bounds on the isoperimetric function and the distortion of a rewriting process do not depend up to $\cong$-equivalence on a given presentation.

If a rewriting process ρ minimises the word length, i.e. $|\rho(w)| = \min\{|v| \text{ for } v \in E$ and $v =_G w\}$ for all $w \in H$, then δ_ρ is called the *distortion of H in G*, c.f. section 1.2. Analogously, if ρ minimises the area, i.e. $\Delta_P(w^{-1}\rho(w)) = \min\{\Delta_P(w^{-1}v)$ for $v \in E$ and $v =_G w\}$ for all $w \in H$, then Φ_ρ is called the *generalised isoperimetric function of H in G*, cf. [Far94].

1.4 Main Results

Let G be a suitable amalgamation $G_1 *_H G_2$ with G_i for $i = 1, 2$ a finitely presented nilpotent group. Using bracketings introduced in section 2 we construct in section 3 and section 4 a rewriting process ρ from G to H such that ρ has a polynomial upper bound on its distortion and isoperimetric function. Since H is also a nilpotent group, Φ_H has a polynomial upper bound as well. Thus $\Phi_G(n) \preceq \Phi_\rho(n) + \Phi_H(\delta_\rho(n))$ is then bounded above by a polynomial. However, this rewriting process ρ requires a suitable central series for G_1 and G_2.

Let $\iota_i : H \to G_i$ be the injection of H in G_i. We call the amalgamation G a *double* and denote it by $G_1 *_{H,id} G_2$, if and only if $G_1 = G_2$ and $\iota_1 = \iota_2$. In section 5 we show that for doubles suitable central series of the form required by the rewriting process ρ in section 4 exist. Thereby we get:

Theorem 2 *Let G be a* double *of a finitely generated nilpotent group of class c. Then*

$$\Phi_G(n) \preceq n^{2c^2}.$$

We denote the j-th term of the lower central series of G_i by $\gamma_j G_i$. In section 6 we introduce non-twisted amalgamations:

Definition 1 *We call an amalgamation $G_1 *_H G_2$* non-twisted *if and only if*

$$\gamma_j G_1 \cap H \not\subseteq \gamma_k G_2 \quad \textit{implies} \quad \gamma_k G_2 \cap H \subseteq \gamma_j G_1$$

for all $j, k \in N$.

If $G_1 *_H G_2$ is non-twisted then $\gamma_k G_2 \cap H \not\subseteq \gamma_j G_1$ also implies $\gamma_j G_1 \cap H \subseteq \gamma_k G_2$, c.f. lemma 4. We note that doubles are examples of non-twisted amalgamations. We show that if G is non-twisted then there exist central series for G_1 and G_2 of the form required by the rewriting process ρ constructed in section 4. Thereby we get our main result:

Theorem 4 *Let G be a* non-twisted *amalgamation of finitely generated* nilpotent *groups of class c. Then*

$$\Phi_G(n) \preceq n^{2^{(2c+1)}c^2}.$$

In section 7 we focus on amalgamations along abelian subgroups:

Theorem 5 *Let $G = G_1 *_H G_2$ be an amalgamation of finitely generated* nilpotent *groups. Suppose*

$$H \subseteq \gamma_j G_q \quad \text{and} \quad H \cap \gamma_{j+1} G_q = \{1\}$$

for some positive integer j and $q = 1$ or $q = 2$. Hence H is abelian. Then G is non-twisted *and thereby satisfies a* polynomial isoperimetric inequality.

We conclude this section by giving an example of a twisted, i.e. not non-twisted, amalgamation along an abelian subgroup having an exponential isoperimetric function:

Theorem 7 *Let G_i for $i = 1, 2$ be the free nilpotent group of class 2 and rank 2. There exists a twisted amalgamation $G = G_1 *_H G_2$ with H* abelian, isolated *and* normal *such that*

$$\Phi_G(n) \cong 2^n.$$

2 Bracketings

Given a suitable presentation for an amalgamation, we introduce bracketings for words representing elements in the amalgamated subgroup. We will use bracketings in the following sections to construct rewriting processes from an amalgamation to its amalgamated subgroup.

Let F be the free group freely generated by $\mathcal{F}_1 \cup \mathcal{F}_2$. A product of words $v_0 \cdots v_t \in F$ is called an *alternating product* if and only if $v_j \in F_{i_j}$ and $v_{j+1} \notin F_{i_j}$ for $1 \leq j < t$ and for $t > 0$ all v_j are not empty. In this case t is called *the number of alternations in $v_0 \cdots v_t$*. Clearly any word in F can be written as an alternating product.

Let $G = G_1 *_H G_2$ where G_i for $i = 1, 2$ is a finitely presented group, and H is a finitely generated subgroup of G_i. Let $w = v_0 \cdots v_t \in H$ be an alternating product. Thus $v_j \in H$ for some j. In the following definition we define a bracketing for w such that any subword of w enclosed by brackets represents an element of H. By lemma 1 a bracketing exists for a word w if and only if $w \in H$.

Definition 2 *Suppose $G = G_1 *_H G_2$ where G_i for $i = 1, 2$ is generated by $\mathcal{F}_i$, and H is a subgroup of G_i. We define bracketings for some words w in the generators $\mathcal{F}_1 \cup \mathcal{F}_2$ by induction on the number t of alternations in w.*

1. If $t = 0$ then (w) is a bracketing *for w if and only if $w \in H$.*

Suppose we have defined bracketings for alternating products with less than t alternations and $w = v_0 \cdots v_t$ is an alternating product.

2. If for some $j < t$ the two alternating products $v_0 \cdots v_j$ and $v_{j+1} \cdots v_t$ have bracketings β_1 and β_2 then $\beta_1 \beta_2$ is a bracketing for w.

3. If $w = v_0 w_0 v_{j_1} w_1 \cdots v_{j_l} w_l v_t \in H$ such that for each w_i $(0 \le i \le l)$ there exists a bracketing β_i and $v_{j_i} w_i \cdots v_{j_k} w_k$ is not in H for $0 \le i \le k \le l$ and $j_0 = 0$ then $(v_0 \beta_1 v_{j_1} \cdots v_{j_l} \beta_l v_t)$ is a bracketing for w.

Lemma 1 *[Hid97, section 4] Suppose $w \in G_1 *_H G_2$ is a word in the generators of G_1 and G_2. There exists a bracketing for w if and only if w is an element of H.*

3 Collection

Let $G_1 *_H G_2$ be an amalgamation of finitely generated nilpotent groups and $w \in H$ a word in the generators of G_1 and G_2. To construct in section 4 a rewriting process ρ from $G_1 *_H G_2$ to H we will use rewriting processes ρ_i from the G_i to H for $i = 1, 2$ and induction on the number of alternations of w. In order to rewrite w to a word in the generators of H, we only have to consider, by section 2, the following three cases: *1)* w a word either in the generators of G_1 or a word in the generators of G_2, *2)* $w = w_1 w_2$ for some w_1, $w_2 \in H$ and *3)* $w = v_1 w_1 v_2 \cdots w_k v_{k+1}$ with all $w_j \in H$ and all $v_j \in G_1$ or all $v_j \in G_2$. In this section we focus on the last case. Let all $v_j \in G_i$. Based on lemma 2 we construct in lemma 3 a word x in the generators of G_i such that

$$w = v_1 w_1 v_2 \cdots w_k v_{k+1} =_G w_1 w_2 \cdots w_k x v_{k+1},$$

i.e. we collect the subwords w_j to the left. We then derive upper bounds on the length of x and the area of $w^{-1} w_1 w_2 \cdots w_k x v_{k+1}$.

For convenience we introduce the following convention: For a finite presentation $P = \langle \mathcal{F} \mid \mathcal{R} \rangle$ we denote by F the free group freely generated by $\mathcal{F}$ and by R the normal closure of $\mathcal{R}$ in F. Analogously, if $\mathcal{E}$ is a subset of $\mathcal{F}$ we denote by E the subgroup of F generated by $\mathcal{E}$. If $\mathcal{U}$ is a set of words we denote by $\mathcal{U}^{\pm 1}$ the set $\{u, u^{-1} \mid u \in \mathcal{U}\}$. For a word $w \in F$ we denote the number of letters in $\mathcal{E}$ by $|w|_{\mathcal{E}}$ and call it the *relative length of w with respect to $\mathcal{E}$*. For words $v, w \in F$ we denote by $[v, w]$ the *commutator* $v^{-1} w^{-1} v w$. Let $P = \langle \mathcal{F} \mid \mathcal{R} \rangle$ be a finite presentation for a group G and w, v words in the generators $\mathcal{F}$. By $w = v$ we denote equality in the word-monoid generated by $\mathcal{F}$, by $w =_F v$ equality in the free group F and by $w =_G v$ equality in G.

Let m_j for $j = 1, \ldots, d$ be non-negative integers. By

$$\sum_{\sum_{r=1}^{d} r p_r \le j} m_1^{p_1} \cdots m_d^{p_d}$$

we denote the finite sum of $m_1^{p_1}\cdots m_d^{p_d}$ over all d-tuples $(p_1,\ldots,p_d)$ of non-negative integers p_r such that $\sum_{r=1}^d r\,p_r \le j$.

Lemma 2 *Let G be a nilpotent group finitely presented by $P=\langle\mathcal{F}\,|\,\mathcal{R}\rangle$ and*

$$G=N_1\supseteq N_2\supseteq\ldots\supseteq N_d\supseteq N_{d+1}=\{1\}$$

a central series of G such that $[N_i,N_j]\subseteq N_{i+j}$. Suppose $\mathcal{F}$ is the disjoint union of $\mathcal{N}_i$ for $i=1,\ldots,d$ such that $\mathcal{N}_i$ generates N_i. There exists a map $\eta:\mathcal{F}\times F\to F$ and a positive integer D such that

$$\eta(e,w)=_G w^{-1}ew \quad \text{and} \quad \eta(e,w)\in F_i \qquad \text{for } e\in\mathcal{N}_i,\ w\in F.$$

Moreover, $\eta(e,w)$ satisfies the following inequalities

$$|\eta(e,w)|_{\mathcal{N}_j} \le D^{j-i}\sum_{\sum_{r=1}^j rp_r\le j-i} n_1^{p_1}\cdots n_d^{p_d} \qquad (1)$$

$$\Delta_P(w^{-1}e^{-1}w\eta(e,w)) \le D^{2d-i}\sum_{\sum_{r=1}^j rp_r\le 2d-i} n_1^{p_1}\cdots n_d^{p_d} \qquad (2)$$

with $n_j=|w|_{\mathcal{N}_j}$ for $j=1,\ldots,d$.

Proof: Let $w\in F$, $n=|w|$ and $e\in\mathcal{F}^{\pm1}$. We define $\eta(e,w)$ by induction on n.

For $n=0$ we have $w=1$. Hence we define $\eta(e,1)$ by e.

Suppose $n>0$ and η is defined for all words of length $<n$. Let $w\in F$ such that $|w|=n$. Let $e\in\mathcal{N}_i$, $w=fv$ with $f\in\mathcal{N}_k$ and $v\in F$. Let $u_{e,f}=g_1\cdots g_t$ with $g_s\in\mathcal{N}_{i+k}$ for $s=1,\ldots,t$. By the induction hypothesis we have

$$\begin{aligned} w^{-1}ew &=_F v^{-1}ee^{-1}f^{-1}efv=_G v^{-1}eu_{e,f}v=v^{-1}eg_1\cdots g_tv\\ &=_G \eta(e,v)\eta(g_1,v)\cdots\eta(g_t,v).\end{aligned}$$

We define $\eta(e,w)$ by $\eta(e,v)\eta(g_1,v)\cdots\eta(g_t,v)$.

Since $[N_i,N_j]\subseteq N_{i+j}$ there exists, for any letter $e\in\mathcal{N}_i$ and $f\in\mathcal{N}_k$, a word $u_{e,f}$ in the generators $\mathcal{N}_{i+k}$ such that $u_{e,f}=_G[e,f]$. Let D be a positive integer such that $|u_{e,f}|$ and $\Delta_P([e,f]^{-1}u_{e,f})\le D$ for all e,$f\in\mathcal{F}^{\pm1}$. We prove the inequalities (1) and (2) by induction on n.

For $n=0$ the inequalities hold.

Suppose $n>0$ and the inequalities hold for all words v of length $<n$. Let $e\in\mathcal{N}_i^{\pm1}$, $w\in F$ and $f\in\mathcal{N}_k^{\pm1}$ such that $|w|=n$ and $w=fv$. Let $u_{e,f}=g_1\cdots g_t$ with $g_s\in\mathcal{N}_{i+k}^{\pm1}$ for $s=1,\ldots,t$. By the definition of $\eta(e,w)$ we have

$$|\eta(e,w)|_{\mathcal{N}_j}=|\eta(e,fv)|_{\mathcal{N}_j}\le|\eta(e,v)|_{\mathcal{N}_j}+\textstyle\sum_{s=1}^t|\eta(g_s,v)|_{\mathcal{N}_j}.$$

Let $n_j=|w|_{\mathcal{N}_j}$ for $j=1,\ldots,d$. Since $|v|_{\mathcal{N}_k}=n_k-1$ and $|v|_{\mathcal{N}_j}=n_j$ for $j\ne k$ we get by $e\in\mathcal{N}_i$, $g_s\in\mathcal{N}_{i+k}$ and the induction hypothesis

$$\begin{aligned}|\eta(e,w)|_{\mathcal{N}_j} \le\ & D^{j-i}\sum_{\sum_{r=1}^d rp_r\le j-i} n_1^{p_1}\cdots n_{k-1}^{p_{k-1}}(n_k-1)^{p_k}n_{k+1}^{p_{k+1}}\cdots n_d^{p_d}+\\ & D^{j-i-k}\sum_{s=1}^t\sum_{\sum_{r=1}^d rq_r\le j-i-k} n_1^{q_1}\cdots(n_k-1)^{q_k}\cdots n_d^{q_d}.\end{aligned}$$

Since $t \leq D$ and $k \geq 1$ we have

$$
\begin{aligned}
& D^{j-i-k} \sum_{s=1}^{t} \sum_{\sum_{r=1}^{d} r q_r \leq j-i-k} n_1^{q_1} \cdots (n_k-1)^{q_k} \cdots n_d^{q_d} \\
& \leq \; D^{j-i} \sum_{\substack{\sum_{r=1}^{d} r q_r \leq j-i \\ q_k \geq 1}} n_1^{q_1} \cdots (n_k-1)^{q_k-1} \cdots n_d^{q_d} \qquad (3)
\end{aligned}
$$

because if a d-tupel $(q_1,\ldots,q_d)$ satisfies $\sum_{r=1}^{d} r q_r \leq j-i-k$ then $(q_1,\ldots,q_k+1,\ldots,q_d)$ satisfies $\sum_{r=1}^{d} r q_r \leq j-i$. Thus we get

$$
\begin{aligned}
|\eta(e,w)|_{\mathcal{N}_j} \; \leq \; & D^{j-i} \sum_{\substack{\sum_{r=1}^{d} r p_r \leq j-i \\ p_k = 0}} n_1^{p_1} \cdots (n_k-1)^{p_k} \cdots n_d^{p_d} + \\
& D^{j-i}(n_k-1) \sum_{\substack{\sum_{r=1}^{d} r p_r \leq j-i \\ p_k \geq 1}} n_1^{p_1} \cdots (n_k-1)^{p_k-1} \cdots n_d^{p_d} + \\
& D^{j-i} \sum_{\substack{\sum_{r=1}^{d} r q_r \leq j-i \\ q_k \geq 1}} n_1^{q_1} \cdots (n_k-1)^{q_k-1} \cdots n_d^{q_d} \\
\leq \; & D^{j-i} \sum_{\sum_{r=1}^{d} r p_r \leq j-i} n_1^{q_1} \cdots n_k^{p_k} \cdots n_d^{p_d}.
\end{aligned}
$$

Hence inequality (1) holds.

We note that if $x^{-1}yx =_G 1$ then $\Delta_P(x^{-1}yx) = \Delta_P(y)$ and if $xy =_G 1 =_G y^{-1}z$ then $\Delta_P(xz) \leq \Delta_P(xy) + \Delta_P(y^{-1}z)$. As above, let $\eta(e,w) = g_1 \cdots g_t =_G w^{-1}ew$ with $g_s \in \mathcal{N}_{i+k}^{\pm 1}$ for $s = 1,\ldots,t$ and $t \leq D$. Thus we get

$$
\begin{aligned}
& \Delta_P(w^{-1}e^{-1}w\eta(e,w)) \\
& \quad \leq \; \Delta_P(v^{-1}f^{-1}e^{-1}feu_{e,f}v) + \Delta_P(v^{-1}u_{e,f}^{-1}e^{-1}v\eta(e,w)) \\
& \quad \leq \; D + \Delta_P(v^{-1}g_t^{-1}\cdots g_1^{-1}e^{-1}v\eta(e,v)\eta(g_1,v)\cdots\eta(g_t,v)) \\
& \quad \leq \; D + \Delta_P(v^{-1}g_t^{-1}\cdots g_1^{-1}vv^{-1}e^{-1}v\eta(e,v)v^{-1}g_1\cdots g_t v) + \\
& \qquad\;\; \Delta_P(v^{-1}g_t^{-1}\cdots g_1^{-1}v\eta(g_1,v)\cdots\eta(g_t,v)).
\end{aligned}
$$

Since $\eta(g_s,v) =_G v^{-1}g_s v$ we get

$\Delta_P(w^{-1}e^{-1}w\eta(e,w)) \leq +\Delta_P(v^{-1}e^{-1}v\eta(e,v)) + \sum_{s=1}^{t} \Delta_P(v^{-1}g_s^{-1}v\eta(g_s,v))$.

Because $|v|_{\mathcal{N}_k} = n_k - 1$ and $|v|_{\mathcal{N}_j} = n_j$ for $j \neq k$ we get by the induction hypothesis

$$
\begin{aligned}
\Delta_P(w^{-1}e^{-1}w\eta(e,w)) \; \leq \; & D + D^{2d-i} \sum_{\sum_{r=1}^{d} r p_r \leq 2d-i} n_1^{p_1} \cdots (n_k-1)^{p_k} \cdots n_d^{p_d} + \\
& D^{2d-i-k} \sum_{s=1}^{t} \sum_{\sum_{r=1}^{d} r q_r \leq 2d-i-k} n_1^{q_1} \cdots (n_k-1)^{q_k} \cdots n_d^{q_d}.
\end{aligned}
$$

By $t \leq D$ we get as in (3)

$$\begin{aligned}
&\Delta_P(w^{-1}e^{-1}w\eta(e,w)) \\
\leq\ & D + D^{2d-i} \sum_{\substack{\sum_{r=1}^{d} rp_r \leq 2d-i \\ p_k=0}} n_1^{p_1}\cdots(n_k-1)^{p_k}\cdots n_d^{p_d} + \\
& D^{2d-i}(n_k-1) \sum_{\substack{\sum_{r=1}^{d} rp_r \leq 2d-i \\ p_k\geq 1}} n_1^{p_1}\cdots(n_k-1)^{p_k-1}\cdots n_d^{p_d} + \\
& D^{2d-i} \sum_{\substack{\sum_{r=1}^{d} rq_r \leq 2d-i \\ q_k\geq 1}} n_1^{q_1}\cdots(n_k-1)^{q_k-1}\cdots n_d^{q_d} \\
\leq\ & D^{2d-i} \sum_{\sum_{r=1}^{d} rp_r \leq 2d-i} n_1^{q_1}\cdots n_k^{p_k}\cdots n_d^{p_d}.
\end{aligned}$$

Hence inequality (2) holds as well. □

Lemma 3 *Let G be a nilpotent group finitely presented by $P = \langle \mathcal{F} \,|\, \mathcal{R} \rangle$ and*

$$G = N_1 \supseteq N_2 \supseteq \ldots \supseteq N_d \supseteq N_{d+1} = \{1\}$$

a central series of G such that $[N_i, N_j] \subseteq N_{i+j}$. Suppose $\mathcal{F}$ is the disjoint union of $\mathcal{N}_i$ for $i = 1, \ldots, d$ such that $\mathcal{N}_i$ generates N_i. For $s = 1, \ldots, t$ let v_s be a word in the generators $\mathcal{N}_1$, $n = \sum_{s=1}^{t} |v_s|$ and $w_s \in F$ such that $\sum_{s=1}^{t} |w_s|_{\mathcal{N}_j} \leq m^j$ for a positive integer m. There exists a word $x \in F$ such that $w_1 w_2 \cdots w_t x =_G v_1 w_1 v_2 w_2 \ldots v_t w_t$,

$$|x|_{\mathcal{N}_j} \leq K_j\, n\, m^{j-1} \qquad \textit{and} \tag{4}$$

$$\Delta_P((v_1 w_1 \cdots v_t w_t)^{-1} w_1 \cdots w_t x) \leq A\, n\, m^{2d-1} \tag{5}$$

with A and K_j for $j = 1, \ldots, d$ suitable positive integers.

Proof: By lemma 2 there exists a map $\eta : N_1 \times F \to F$ and a positive integer D such that $\eta(v, w) =_G w^{-1}vw$ for v a word in the generators $\mathcal{N}_1$, $w \in F$ and

$$|\eta(v,w)|_{\mathcal{N}_j} \leq |v| D^{j-1} \sum_{\sum_{r=1}^{j} rp_r \leq j-1} |w|_{\mathcal{N}_1}^{p_1} \cdots |w|_{\mathcal{N}_d}^{p_d}\ , \tag{6}$$

$$\Delta_P(w^{-1}v^{-1}w\eta(v,w)) \leq |v| D^{2d-1} \sum_{\sum_{r=1}^{j} rp_r \leq 2d-1} |w|_{\mathcal{N}_1}^{p_1} \cdots |w|_{\mathcal{N}_d}^{p_d}. \tag{7}$$

Let v_s for $s = 1, \ldots, t$ be a word in the generators $\mathcal{N}_1$, $n = \sum_{s=1}^{t} |v_s|$ and $w_s \in F$ such that $\sum_{s=1}^{t} |w_s|_{\mathcal{N}_j} \leq m^j$ for a positive integer m. By induction on t we define a word x such that (4) holds.

For $t = 1$ we define x by $\eta(v_1, w_1)$ since $v_1 w_1 =_G w_1 \eta(v_1, w_1)$.

Suppose $t > 1$ and a word $y \in F$ exists such that

$$w_2 w_3 \cdots w_t y =_G v_2 w_2 v_3 w_3 \ldots v_t w_t$$

and y satisfies (4). Hence

$$\begin{aligned} v_1 w_1 v_2 w_2 \cdots v_t w_t &=_G v_1 w_1 w_2 w_3 \cdots w_t y \\ &=_G w_1 w_2 \cdots w_t \eta(v_1, w_1 w_2 \cdots w_t) y. \end{aligned}$$

We define x by $\eta(v_1, w_1 w_2 \cdots w_t) y$.

We prove inequality (4) by induction on t.

For $t = 1$ inequality (6) yields

$$\begin{aligned} |x|_{\mathcal{N}_j} = |\eta(v_1, w_1)|_{\mathcal{N}_j} &\leq nD^{j-1} \sum_{\sum_{r=1}^{j} r p_r \leq j-1} m^{1 \cdot p_1} \cdots m^{d \cdot p_d} \\ &\leq nD^{j-1} \sum_{\sum_{r=1}^{j} r p_r \leq j-1} m^{j-1} \leq D^{j-1} j^{j-1} n m^{j-1}. \end{aligned}$$

We define K_j by $D^{j-1} j^{j-1}$. Thus inequality (4) holds for $t = 1$.

Suppose $t > 1$ and (4) holds for y. By $|w_1 w_2 \cdots w_t|_{\mathcal{N}_j} \leq m^j$ and inequality (6) we get

$$\begin{aligned} |x|_{\mathcal{N}_j} &\leq |\eta(v_1, w_1 w_2 \cdots w_t)|_{\mathcal{N}_j} + |y|_{\mathcal{N}_j} \\ &\leq |v_1| D^{j-1} \sum_{\sum_{r=1}^{j} r p_r \leq j-1} m^{1 \cdot p_1} \cdots m^{d \cdot p_d} + K_j (n - |v_1|) m^{j-1} \\ &\leq |v_1| D^{j-1} \sum_{\sum_{r=1}^{j} r p_r \leq j-1} m^{j-1} + K_j (n - |v_1|) m^{j-1} \\ &\leq |v_1| D^{j-1} j^{j-1} m^{j-1} + K_j (n - |v_1|) m^{j-1} \leq K_j \, n \, m^{j-1}. \end{aligned}$$

Thus inequality (4) holds.

We prove inequality (5) by induction on t. Let $A = D^{2d-1} (2d)^{2d-1}$.

For $t = 1$ inequality (5) holds by (7).

Suppose $t > 1$ and (5) holds for y. Since

$$\begin{aligned} &\Delta_P((v_1 w_1 v_2 w_2 \cdots v_t w_t)^{-1} w_1 w_2 \cdots w_t x) \\ &\quad \leq \Delta_P((v_1 w_1 v_2 w_2 \cdots v_t w_t)^{-1} v_1 w_1 w_2 w_3 \cdots w_t y) + \\ &\qquad \Delta_P((v_1 w_1 w_2 w_3 \cdots w_t y)^{-1} w_1 w_2 \cdots w_t \eta(v_1, w_1 w_2 \cdots w_t) y) \\ &\quad \leq \Delta_P((v_2 w_2 v_3 w_3 \cdots v_t w_t)^{-1} w_2 w_3 \cdots w_t y) + \\ &\qquad \Delta_P((w_1 w_2 \cdots w_t)^{-1} v_1^{-1} w_1 w_2 w_3 \cdots w_t \eta(v_1, w_1 w_2 \cdots w_t)) \end{aligned}$$

we get by the induction hypothesis and (7)

$$\begin{aligned} \Delta_P((v_1 w_1 v_2 w_2 \cdots v_t w_t)^{-1} w_1 w_2 \cdots w_t x) &\leq A(n - |v_1|) m^{2d-1} + A |v_1| m^{2d-1} \\ &\leq A \, n \, m^{2d-1}. \end{aligned}$$

Hence inequality (5) holds as well. □

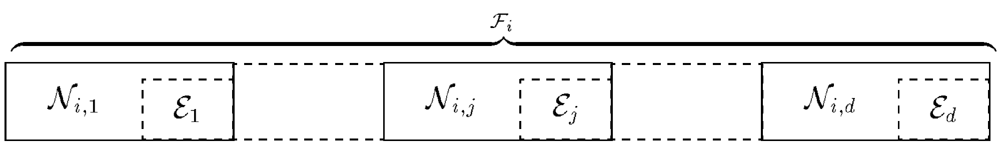
4 Rewriting along a Central Series

Let $G = G_1 *_H G_2$ be an amalgamation of finitely presented nilpotent groups. Suppose there exists a central series $N_{i,j}$ for $i = 1, 2$ of length $< d$ for G_i such that

$$N_{1,j} \cap H = N_{2,j} \cap H \quad \text{and} \quad [N_{i,r}, N_{i,s}] \subseteq N_{i,r+s} \tag{8}$$

for all positive integers r and s. The goal of this section is to construct in proposition 1 a rewriting process ρ from G to H and to derive upper bounds on the distortion and the isoperimetric function of ρ. We show in the following sections that central series of the form (8) exist for doubles, non-twisted amalgamations and some amalgamations along abelian subgroups.

We give an outline of the construction of ρ: Let $P_i = \langle \mathcal{F}_i \,|\, \mathcal{R}_i \rangle$ be a finite presentation for G_i of the following form (see the figure below): $\mathcal{F}_i$ is the disjoint union of $\mathcal{N}_{i,j}$ for $j = 1, \ldots, d$ such that $\mathcal{N}_{i,j}$ generates $N_{i,j}$. Since $N_{1,j} \cap H = N_{2,j} \cap H$ each $\mathcal{N}_{i,j}$ contains a subset $\mathcal{E}_j$ generating $H \cap N_{i,j}$ and $\mathcal{E} = \bigcup_{j=1}^{d} \mathcal{E}_j$ generates H.



Hence there exits a finite presentation $P = \langle \mathcal{F} \,|\, \mathcal{R} \rangle$ for G with $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$, $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ such that $\mathcal{E} = \mathcal{F}_1 \cap \mathcal{F}_2$ and $\mathcal{E}_j = \mathcal{N}_{1,j} \cap \mathcal{N}_{2,j}$. Let ρ_i be a rewriting process from G_i to H relative to P_i, $\mathcal{E}$. Let $w \in F$ represent an element in H. Using ρ_i we construct $\rho(w)$, our rewriting process ρ from G to H relative to P, $\mathcal{E}$, by induction on the number of alternations of w.

Suppose w has no alternations: Thus w is an element of F_i for $i = 1$ or $i = 2$. We define $\rho(w)$ as $\rho_i(w)$, an element in E.

Suppose w has $t \geq 1$ alternations: Since w represents an element in H there exists by section 2, lemma 1 a bracketing β for w. If β is of the form $\beta_1\beta_2$, then $w = w_1w_2$ with w_1, w_2 representing words in H having $< t$ alternations. Hence we define $\rho(w)$ as $\rho(w_1)\rho(w_2)$ by the induction hypothesis. If β is of the form $(v_1\beta_1v_2 \cdots \beta_kv_{k+1})$, then $w = v_1w_1v_2 \cdots w_kv_{k+1}$ with all $v_j \in F_1$ or all $v_j \in F_2$ and all $w_j \in H$ having $< t$ alternations. Let all $v_j \in F_i$. By section 3, lemma 3 there exists a word $x \in F_i$ such that $w =_G w_1w_2 \cdots w_kxv_{k+1}$. Since all w_j represent elements in H, the word $xv_{k+1} \in F_i$ is also an element of H. We define $\rho(w)$ as $\rho(w_1)\rho(w_2) \cdots \rho(w_k)\rho_i(xv_{k+1})$, a word in the generators of H. To compute upper bounds on the distortion and the isoperimetric function of ρ we use the corresponding upper bounds on x given by lemma 3 and on ρ_i given by the following theorem. It is crucial for the proof of proposition 1 to express these upper bounds in terms of $|\cdot|_{\mathcal{N}_{i,j}}$, the relative length with respect to $\mathcal{N}_{i,j}$, and not in the full wordlength $|\cdot|$.

Theorem 1 *[Hid97, Section 3.2] Let G be a finitely presented nilpotent group, H a subgroup of G and let*

$$G = N_1 \supseteq N_2 \supseteq \ldots \supseteq N_d \supseteq N_{d+1} = \{1\}$$

be a central series of G such that $[N_r, N_s] \subseteq N_{r+s}$ for all positive integers r and s.

- *There exists a finite presentation $P = \langle \mathcal{F} \,|\, \mathcal{R} \rangle$ for G such that $\mathcal{F}$ is the disjoint union of $\mathcal{N}_j$ for $j = 1, \ldots, d$ and $\mathcal{N}_j$ generates N_j. Each $\mathcal{N}_j$ contains a subset $\mathcal{E}_j$, which generates $H \cap N_j$.*
- *Let $\mathcal{E} = \bigcup_{j=1}^{d} \mathcal{E}_j$. Thus $\mathcal{E}$ generates H. There exists a rewriting process ρ from G to H relative to P, $\mathcal{E}$ and a positive integer K such that for $j = 1, \ldots, d$*

$$|\rho(w)|_{\mathcal{N}_j} \leq K \sum_{\sum_{r=1}^{d} r p_r \leq j} n_1^{p_1} \cdots n_d^{p_d} \tag{9}$$

and

$$\Delta_P(w^{-1}\rho(w)) \leq K \sum_{\sum_{r=1}^{d} r p_r \leq 2d} n_1^{p_1} \cdots n_d^{p_d} \tag{10}$$

with $w \in H$ and $n_j = \sum_{k=1}^{j} |w|_{\mathcal{N}_k}$.

Remark 1 *We note that for G a finitely generated nilpotent group of class c theorem 1 implies $\Phi_G(n) \preceq n^{2c}$: Let $N_j = \gamma_j G$ for $j = 1, \ldots, c+1$. Hence $N_{c+1} = \{1\}$ and $[N_r, N_s] \subseteq N_{r+s}$ for all r and s. By theorem 1 there exists a rewriting process ρ from G to $\{1\}$ with respect to some finite presentation P such that*

$$\Delta_P(w^{-1}\rho(w)) \leq K \sum_{\sum_{r=1}^{c} r p_r \leq 2c} n^{p_1} \cdots n^{p_c}$$

for all words $w =_G 1$, $n = |w|$ and some constant K. Hence

$$\Delta_P(w^{-1}\rho(w)) \leq K L n^{2c}$$

for some constant L, yielding $\Phi_G(n) \preceq n^{2c}$, c.f. [Hid97].

Proposition 1 *Let $G = G_1 *_H G_2$ where G_i for $i = 1, 2$ is a finitely presented nilpotent group and H a subgroup of G_i. Suppose*

$$G_i = N_{i,1} \supseteq N_{i,2} \supseteq \ldots \supseteq N_{i,d} \supseteq N_{d+1} = \{1\}$$

a central series of G_i such that $[N_{i,r}, N_{i,s}] \subseteq N_{i,r+s}$ and $N_{1,j} \cap H = N_{2,j} \cap H$ for $j, r, s = 1, \ldots, d$. There exists a rewriting process ρ from G to H such that

$$\delta_\rho(n) \preceq n^d \quad \text{and} \quad \Phi_\rho(n) \preceq n^{2d+1}.$$

Proof: Let $P_i = \langle \mathcal{F}_i \,|\, \mathcal{R}_i \rangle$ for $i = 1, 2$ be the finite presentation for G_i given by theorem 1. Let $P = \langle \mathcal{F} \,|\, \mathcal{R} \rangle$ with $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$, $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ be a finite presentation for G such that $\mathcal{E} = \mathcal{F}_1 \cap \mathcal{F}_2$ generates H. Since $N_{1,j} \cap H = N_{2,j} \cap H$ we may assume, without loss of generality, that $\mathcal{E}_j = \mathcal{N}_{1,j} \cap \mathcal{N}_{2,j}$ generates $N_{1,j} \cap H = N_{2,j} \cap H$ and

$\mathcal{E} = \bigcup_{j=1}^{d} \mathcal{E}_j$. Let ρ_i be the rewriting process from G_i to H relative to P_i, $\mathcal{E}$ given by theorem 1.

Suppose $w \in H$ is a word in F and β a bracketing for w. We may assume, without loss of generality, that w is a word in the generators $\mathcal{N}_{1,1} \cup \mathcal{N}_{2,1}$. We define $\rho(w)$ by induction on the number t of alternations in w.

For $t = 0$ we have $\beta = (w)$ with $w \in F_i$. We define $\rho(w)$ by $\rho_i(w)$.

Suppose $t > 0$ and ρ is defined for all words with less than t alternations.

Suppose $\beta = \beta_1\beta_2$. Let $w = w_1 w_2$ with β_l for $l = 1, \ldots, k$ a bracketing for w_l. We define $\rho(w)$ by $\rho(w_1)\rho(w_2)$.

Suppose $\beta = (v_1\beta_1 \cdots \beta_k v_{k+1})$. Let $w = v_1 w_1 \cdots w_k v_{k+1}$ with β_l a bracketing for w_l and $v_l \in N_{i,1}$. Because $\rho(w_l)$ are words in $E \subseteq F_i$, there exists by lemma 3 a word $x \in F_i$ such that

$$\rho(w_1) \cdots \rho(w_k) x =_G v_1 \rho(w_1) \cdots v_k \rho(w_k). \tag{11}$$

Hence

$$\begin{aligned} w = v_1 w_1 \cdots w_k v_{k+1} &=_G v_1 \rho(w_1) \cdots v_k \rho(w_k) v_{k+1} \\ &=_G \rho(w_1) \cdots \rho(w_k) x v_{k+1} =_G \rho(w_1) \cdots \rho(w_k) \rho_i(x v_{k+1}). \end{aligned}$$

Since $x v_{k+1} \in H$ is a word in F_i the rewriting process ρ_i is defined on $x v_{k+1}$. We define $\rho(w)$ by $\rho(w_1) \cdots \rho(w_k) \rho_i(x v_{k+1})$.

Let $w \in H$ be a word in the generators $\mathcal{N}_{1,1} \cup \mathcal{N}_{2,1}$, $n = |w|$ and t the number of alternations in w. By induction on t we prove

$$|\rho(w)|_{\mathcal{E}_j} \leq D^j n^j \tag{12}$$

for a suitable positive integer D. This implies $\delta_\rho(n) \preceq n^d$, since $j \leq d$.

For $t = 0$ we have $w \in F_i$ and $\rho(w) = \rho_i(w)$. Since w is a word in the generators $\mathcal{N}_{i,1}$ we have $\sum_{k=1}^{j} |w|_{\mathcal{N}_{i,k}} = n$ for $j = 1, \ldots, d$. Hence we get by theorem 1, inequality (9)

$$|\rho(w)|_{\mathcal{E}_j} = |\rho_i(w)|_{\mathcal{E}_j} \leq L_1 \sum_{\sum_{r=1}^{j} r p_r \leq j} n^{p_1} \cdots n^{p_j} \leq L_1 \sum_{\sum_{r=1}^{j} r p_r \leq j} n^j$$

for some constant L_1.

For any $j \leq d$ the number of j-tuples $(p_1, \ldots, p_j)$ such that $\sum_{r=1}^{j} r p_r \leq j$ is bounded above by a constant. Hence there exists a positive integer L_2 such that

$$\sum_{\sum_{r=1}^{j} r p_r \leq j} n^j \leq L_2 n^j \tag{13}$$

for all $j \leq d$. Let $D = L_1 L_2 L_4$ with L_4 a positive integer which we will construct below. Thus $|\rho(w)|_{\mathcal{E}_j} \leq L_1 L_2 n^j \leq D n^j$ and inequality (12) holds for $t = 0$.

Suppose (12) holds for all words $w \in H$ in the generators $\mathcal{N}_{1,1} \cup \mathcal{N}_{2,1}$ with less than t alternations. Let w be a word with $t \geq 1$ alternations representing an element in H, $n = |w|$ and β a bracketing for w.

If $\beta = \beta_1\beta_2$ then $w = w_1 w_2$ with $w_k \in H$ for $k = 1, 2$. Hence

$$|\rho(w)|_{\mathcal{E}_j} \leq |\rho(w_1)|_{\mathcal{E}_j} + |\rho(w_2)|_{\mathcal{E}_j} \leq D^j|w_1|^j + D^j|w_2|_2^j \leq D^j n^j$$

by the induction hypothesis (12).

If $\beta = (v_1\beta_1 \cdots \beta_k v_{k+1})$ let $w = v_1 w_1 \cdots w_k v_{k+1}$ such that β_l is a bracketing for w_l. Let $m = \sum_{l=1}^k |w_l|$. Since $w_l \in H$ we have $\sum_{l=1}^k |\rho(w_l)|_{\mathcal{E}_j} \leq D^j m^j$ by the induction hypothesis. Let $x \in F_i$ be the word given in (11). By lemma 3 and $\sum_{l=1}^k |v_l| \leq n - m - |v_{k+1}|$ we have

$$\begin{aligned} |xv_{k+1}|_{\mathcal{N}_{i,j}} &\leq |x|_{\mathcal{N}_{i,j}} + |v_{k+1}|_{\mathcal{N}_{i,j}} \leq L_3(n-m-|v_{k+1}|)D^{j-1}m^{j-1} + |v_{k+1}| \\ &\leq L_3 D^{j-1}(n-m)m^{j-1} \end{aligned} \tag{14}$$

for a suitable positive integer L_3. By $\sum_{r=1}^j |xv_{k+1}|_{\mathcal{N}_{i,r}} \leq jL_3D^{j-1}(n-m)m^{j-1}$ and (9) we get

$$|\rho_i(xv_{k+1})|_{\mathcal{E}_j} \leq L_1 \sum_{\sum_{r=1}^j rp_r \leq j} \prod_{q=1}^j \left(qL_3D^{q-1}(n-m)m^{q-1}\right)^{p_q}. \tag{15}$$

By $\sum_{r=1}^j rp_r \leq j \leq d$ and $m < n$ we have

$$\begin{aligned} \prod_{q=1}^j (qL_3)^{p_q} &\leq d^d L_3^d, \\ \prod_{q=1}^j (D^{q-1})^{p_q} &\leq D^{(\sum_{q=1}^j (q-1)p_q)} \leq D^{j-1} \quad \text{and} \\ \prod_{q=1}^j (n-m)^{p_q} m^{(q-1)p_q} &\leq (n-m)^{(\sum_{q=1}^j p_q)} m^{(\sum_{q=1}^j qp_q - \sum_{q=1}^j p_q)} \\ &\leq (n^{\sum_{q=1}^j p_q} - m^{\sum_{q=1}^j p_q}) m^{(j-\sum_{q=1}^j p_q)} \leq n^j - m^j. \end{aligned}$$

Let $L_4 = d^d L_3^d$. Note that L_3 does not depend on x or v_{k+1}. By (15) and (13) we get

$$|\rho_i(xv_{k+1})|_{\mathcal{E}_j} \leq D^{j-1}L_1L_2L_4(n^j - m^j) \leq D^j(n^j - m^j).$$

By the definition of ρ, the induction hypothesis and $\sum_{l=1}^k D^j|w_l|^j \leq D^j m^j$ we get

$$\begin{aligned} |\rho(w)|_{\mathcal{E}_j} &\leq \sum_{l=1}^k |\rho(w_l)|_{\mathcal{E}_j} + |\rho(xv_{k+1})|_{\mathcal{E}_j} \\ &\leq \sum_{l=1}^k D^j|w_l|^j + D^j(n^j - m^j) \leq D^j n^j. \end{aligned}$$

Therefore inequality (12) holds.

Let $w \in H$ be a word in the generators $\mathcal{N}_{1,1} \cup \mathcal{N}_{2,1}$, $n = |w|$ and t the number of alternations in w. By induction on t we prove

$$\Delta_P(w^{-1}\rho(w)) \leq A(t+1)n^{2d} \tag{16}$$

for a suitable positive integer A. By $t+1 \leq n$ we then have $\Phi_\rho(n) \preceq n^{2d+1}$.

For $t=0$ we have $w \in F_i$ and $\rho(w) = \rho_i(w)$. Since w is a word in the generators $\mathcal{N}_{i,1}$ we have $\sum_{k=1}^{j} |w|_{\mathcal{N}_{i,k}} = n$ for $j = 1, \ldots, d$. By (10) we get

$$\begin{aligned} \Delta_P(w^{-1}\rho(w)) &= \Delta_P(w^{-1}\rho_i(w)) \\ &\leq Q_1 \sum_{\sum_{r=1}^{j} r p_r \leq 2d} n^{p_1} \cdots n^{p_d} \leq Q_1 \sum_{\sum_{r=1}^{j} r p_r \leq 2d} n^{2d} \leq Q_2 n^{2d} \end{aligned}$$

with Q_1, Q_2 suitable positive integers as in (13). Let $A = Q_1 Q_2 Q_3 Q_4^{2d} Q_5$ with Q_3, Q_4 and Q_5 positive integers which we will define below. Hence for $t=0$ inequality (16) holds.

Suppose (16) holds for all words in the generators $\mathcal{N}_{1,1} \cup \mathcal{N}_{2,1}$ representing an element in H with less than t alternations. Let w be a word with $t \geq 1$ alternations representing an element in H, $n = |w|$ and β a bracketing for w.

If $\beta = \beta_1\beta_2$ then $w = w_1 w_2$ with $w_k \in H$ for $k = 1, 2$. Let t_k be the number of alternations in w_k. Hence

$$\begin{aligned} \Delta_P(w^{-1}\rho(w)) &\leq \Delta_P(w_1^{-1}\rho(w_1)) + \Delta_P(w_2^{-1}\rho(w_2)) \\ &\leq A(t_1+1)|w_1|^{2d} + A(t_2+1)|w_2|^{2d} \leq A(t+1)n^{2d} \end{aligned}$$

by $(t_1+1) + (t_2+1) \leq t_1 + 1$ and the induction hypothesis (12).

If $\beta = (v_1\beta_1 \cdots \beta_k v_{k+1})$ let $w = v_1 w_1 \cdots w_k v_{k+1}$ such that β_l, for $l = 1, \ldots, k$ is a bracketing for w_l. Let $m = \sum_{l=1}^{k} |w_l|$ and $x \in F_i$ the word given in (11). Since $\sum_{l=1}^{k} |\rho(w_l)|_{\mathcal{E}_j} \leq D^j m^j$ by (12), we get by lemma 3 and $\sum_{l=1}^{k} |v_l| \leq n - m$

$$\begin{aligned} \Delta_P((v_1\rho(w_1)\cdots\rho(w_k)v_{k+1})^{-1}\rho(w_1)\cdots\rho(w_k)x v_{k+1}) &\leq Q_3(n-m)m^{2d-1} \\ &\leq Q_3 n^{2d} \qquad (17) \end{aligned}$$

for a suitable positive integer Q_3. Also by lemma 3 we have

$$\sum_{r=1}^{j} |x v_{k+1}|_{\mathcal{N}_{i,r}} \leq Q_4(n-m)m^{j-1}$$

for a suitable positive integer Q_4. Thus we get by $m < n$ and inequality (10) of theorem 1

$$\begin{aligned} \Delta_P(v_{k+1}^{-1}x^{-1}\rho_i(x v_{k+1})) &\leq Q_1 \sum_{\sum_{r=1}^{d} r p_r \leq 2d} \prod_{l=1}^{d} (Q_4(n-m)m^{l-1})^{p_l} \\ &\leq Q_1 \sum_{\sum_{r=1}^{d} r p_r \leq 2d} Q_4^{2d} n^{2d} \leq Q_1 Q_4^{2d} Q_5 n^{2d} \end{aligned}$$

for a suitable positive integer Q_5. We note that Q_5 as well as Q_3 and Q_4 do net depend on w. By $A = Q_1 Q_2 Q_3 Q_4^{2d} Q_5$ we have

$$\Delta_P(v_{k+1}^{-1}x^{-1}\rho_i(x v_{k+1})) \leq A n^{2d}. \qquad (18)$$

Let t_l be the number of alternations in w_l. By the induction hypothesis and the inequalities (10) of theorem 1, (17) and (18) we get

$$\begin{aligned}\Delta_P(w^{-1}\rho(w)) &\leq \Delta_P(w^{-1}v_1\rho(w_1)\cdots\rho(w_k)v_{k+1}) + \\ &\quad \Delta_P((v_1\rho(w_1)\cdots\rho(w_k)v_{k+1})^{-1}\rho(w_1)\cdots\rho(w_k)xv_{k+1}) + \\ &\quad \Delta_P(v_{k+1}^{-1}x^{-1}\rho_i(xv_{k+1})) \\ &\leq \sum_{l=1}^{k} A(t_l+1)|w_l|^{2d} + Q_3n^{2d} + An^{2d}.\end{aligned}$$

Since $\sum_{l=1}^{k}(t_l+1) = t-1$ we eventually have

$$\Delta_P(w^{-1}\rho(w)) \leq A(t-1)m^{2d} + Q_3n^{2d} + An^{2d} \leq A(t+1)n^{2d}.$$

Thus inequality (16) holds. □

5 Doubles

Let G be a double of a finitely generated nilpotent group A. Hence the lower central series of A already satisfies the condition of proposition 1, yielding:

Theorem 2 *Let G be a* double *of a finitely generated nilpotent group of class c. Then*

$$\Phi_G(n) \preceq n^{2c^2}.$$

Proof: Let A be a finitely generated nilpotent group of class c and H a subgroup of A. Let $G = A *_{H,id} A$. Let $N_{i,j} = \gamma_j A$ for $i = 1, 2$. Hence $N_{i,j}$ is a central series of A of length c with $[N_{i,r}, N_{i,s}] \subseteq N_{i,r+s}$ and $N_{1,j} \cap H =_G N_{2,j} \cap H$. By proposition 1 there exists a rewriting process ρ from G to H such that $\delta_\rho \preceq n^c$ and $\Phi_\rho \preceq n^{2c}$. Since H is also nilpotent of class $\leq c$ we have $\Phi_H(n) \preceq n^{2c}$, cf. remark 1. Thus we get

$$\Phi_G(n) \preceq \Phi_\rho(n) + \Phi_H(\delta_\rho(n)) \preceq n^{2c} + (n^c)^{2c} \preceq n^{2c^2}.$$

□

6 Non-Twisted Amalgamations

We first introduce and illustrate non-twisted amalgamations. We then give in section 6.1 an outline of the proof that non-twisted amalgamations of finitely generated nilpotent groups satisfy a polynomial isoperimetric inequality. After proving preparatory lemmata in the following sections we get in section 6.5, theorem 4 our result.

We recall the definition of non-twisted amalgamations:

Let $G_1 *_H G_2$ be an amalgamation and $\gamma_j G_q$ for $q = 1, 2$ the j-th term of the lower central series of G_q. We call G *non-twisted* if and only if

$$\gamma_i G_1 \cap H \not\subseteq \gamma_j G_2 \text{ implies } \gamma_j G_2 \cap H \subseteq \gamma_i G_1$$

for all $i, j \in N$.

Lemma 4 *Let $G_1 *_H G_2$ be a non-twisted amalgamation. Then*

$$\gamma_j G_2 \cap H \not\subseteq \gamma_i G_1 \quad \text{implies} \quad \gamma_i G_1 \cap H \subseteq \gamma_j G_2$$

for all positive integers i and j.

Proof: Let $G_1 *_H G_2$ be a non-twisted amalgamation and let $\gamma_j G_2 \cap H \not\subseteq \gamma_i G_1$ for some positive integers i and j. Suppose

$$\gamma_i G_1 \cap H \not\subseteq \gamma_j G_2.$$

Thus we have by definition $\gamma_j G_2 \cap H \subseteq \gamma_i G_1$, since $G_1 *_H G_2$ is non-twisted, in contradiction to $\gamma_j G_2 \cap H \not\subseteq \gamma_i G_1$. Hence $\gamma_i G_1 \cap H \subseteq \gamma_j G_2$. □

Let $w \in \gamma_j G_1 \cap H$ be a word in the generators of G_1 such that $w \neq_{G_1} 1$ and let v be a word in the generators of G_2 such that $[w, v] \in H$. There exists a positive integer k such that

$$\gamma_j G_1 \cap H \subseteq \gamma_k G_2 \text{ and } \gamma_j G_1 \cap H \not\subseteq \gamma_{k+1} G_2.$$

Hence $[w, v] \in \gamma_{k+1} G_2 \cap H$, but $[w, v]$ is in general not an element of $\gamma_j G_1 \cap H$ anymore. However, if $G_1 *_H G_2$ is non-twisted then $[w, v]$ is an element of $\gamma_j G_1 \cap H$. Thus, commutators in non-twisted amalgamations "respect" the lower central series of its factors. Note that not all amalgamations are non-twisted, c.f. the example in section 7.

6.1 Outline

Let $G = G_1 *_H G_2$ be a non-twisted amalgamation of finitely generated nilpotent groups. We give an outline of the proof that G satisfies a polynomial isoperimetric inequality: The idea is to construct central series $(N_{q,k})_{k \in N}$ for G_q for $q = 1, 2$ such that

$$N_{1,k} \cap H = N_{2,k} \cap H \quad \text{and} \quad [N_{q,r}, N_{q,s}] \subseteq N_{q,r+s} \tag{19}$$

holds for all positive integers k, r and s. By applying proposition 1 we then get a polynomial upper bound on Φ_G in theorem 4.

We proceed as follows: In section 6.2, lemma 5 we construct a refinement $(N_{q,k})_{k \in N}$ of the lower central series of G_q such that

$$N_{1,k} \cap H = N_{2,k} \cap H \qquad \text{for all } k. \tag{20}$$

In lemma 6 of section 6.3 we show that some condition on $(N_{q,k})_{k \in N}$, i.e. that $(N_{q,k})_{k \in N}$ contains a sufficient number of copies of $\gamma_j G_q$ for each j, implies

$$[N_{q,r}, N_{q,s}] \subseteq N_{q,r+s}.$$

In section 6.4, lemma 7 we refine $(N_{i,k})_{k \in N}$ such that the condition is satisfied for all j less or equal to the minimum of the length of $(N_{1,k})_{k \in N}$ and $(N_{2,k})_{k \in N}$, while preserving property (20). In lemma 8 we further refine the resulting central series such that the condition is satisfied for all k. Thereby we get in section 6.5, proposition 2 a refinement $(N_{q,k})_{k \in N}$ of the lower central series of G_q satisfying (19). We then get in theorem 3 by section 4, proposition 1 a rewriting process from G to H having a polynomial upper bound on its distortion and isoperimetric function. Since H is also a nilpotent group, H satisfies polynomial isoperimetric inequality as well. Hence we get in theorem 4 a polynomial upper bound on the isoperimetric function of G.

6.2 Intersection

Let $A *_H B$ be a non-twisted amalgamation of nilpotent groups. In lemma 5 we construct refinments $(A_k)_{k\in N}$, $(B_k)_{k\in N}$ of the lower central series of A and B where A_k is of the form

$$(\gamma_i A \cap \gamma_j B)\gamma_{i+1} A$$

for some positive integers i and j and B_k is of the form $(\gamma_j B \cap \gamma_i A)\gamma_{j+1} B$ for some i and j. Exploiting the non-twistedness of $A *_H B$ we show that

$$A_k \cap H = B_k \cap H$$

for all k.

Lemma 5 *Let $A *_H B$ be a non-twisted amalgamation of finitely generated nilpotent groups of class $\leq c$. There exist refinements $(A_k)_{k\in N}$ and $(B_k)_{k\in N}$ of length $\leq 2c$ of the lower central series of A and B such that*

$$A_k \cap H = B_k \cap H \qquad \textit{for all } k \in N.$$

Proof: The proof proceeds in 3 steps. In step *1)* we define $(A_k)_{k\in N}$ and $(B_k)_{k\in N}$. In step *2)* we show that they refine the lower central series of A and B respectively and in step *3)* that $A_k \cap H = B_k \cap H$ for all $k \geq 1$.

1) We define $(A_k)_{k\in N}$ and $(B_k)_{k\in N}$ by induction on k. For $k = 1$ let

$$A_k = A, \quad B_k = B, \quad i_k = 1, \quad \text{and} \quad j_k = 1. \tag{21}$$

Suppose $k > 1$ and we have defined A_l, B_l, i_l and j_l for all $l < k$. We define A_k, B_k, i_k and j_k as follows:

$$A_k = (\gamma_{i_{k-1}} A \cap \gamma_{j_{k-1}+1} B)\gamma_{i_{k-1}+1} A; \tag{22}$$
$$B_k = (\gamma_{i_{k-1}+1} A \cap \gamma_{j_{k-1}} B)\gamma_{j_{k-1}+1} B; \tag{23}$$
$$\text{if } \gamma_{i_{k-1}+1} A \cap H \subset \gamma_{j_{k-1}+1} B \cap H \text{ then } i_k = i_{k-1},\ j_k = j_{k-1}+1; \tag{24}$$
$$\text{if } \gamma_{j_{k-1}+1} B \cap H \subset \gamma_{i_{k-1}+1} A \cap H \text{ then } i_k = i_{k-1}+1,\ j_k = j_{k-1}; \tag{25}$$
$$\text{if } \gamma_{i_{k-1}+1} A \cap H = \gamma_{j_{k-1}+1} B \cap H \text{ then } i_k = i_{k-1}+1,\ j_k = j_{k-1}+1; \tag{26}$$

Since $A *_H B$ is non-twisted either condition (24), (25) or (26) holds always. Hence we have

$$\begin{gathered} i_{k-1}+1 \geq i_k \geq i_{k-1}, \quad j_{k-1}+1 \geq j_k \geq j_{k-1} \quad \text{and} \\ i_k + j_k > i_{k-1} + j_{k-1}, \quad i_k + j_k \geq k \quad \text{for all } k > 1. \end{gathered} \tag{27}$$

2) In order to prove that $(A_k)_{k\in N}$ and $(B_k)_{k\in N}$ are refinements of length $\leq 2c$ of the lower central series of A and B it suffices to show that *i)* $A_k \subseteq A_{k-1}$, $B_k \subseteq B_{k-1}$ for all $k > 1$, that *ii)* $A_{2c+1} = \{1\} = B_{2c+1}$ and that *iii)* for each j there exist indices m and n such that $\gamma_j A = A_m$ and $\gamma_j B = B_n$.

i) $A_k \subseteq A_{k-1}$ for $k > 1$: For $k = 2$ we have $A_2 \subseteq A = A_1$ by (21). For $k > 2$ we have $i_{k-1} \geq i_{k-2}$ and $j_{k-1} \geq j_{k-2}$ by (27). Hence we get by (22)

$$A_k = (\gamma_{i_{k-1}} A \cap \gamma_{j_{k-1}+1} B)\gamma_{i_{k-1}+1} A \subseteq (\gamma_{i_{k-2}} A \cap \gamma_{j_{k-2}+1} B)\gamma_{i_{k-2}+1} A = A_{k-1}.$$

$B_k \subseteq B_{k-1}$ for all $k > 1$: follows analogously.

ii) $A_{2c+1} = \{1\} = B_{2c+1}$: Since $i_{2c} + j_{2c} \geq 2c$ by (27) we have $i_{2c} \geq c$ or $j_{2c} \geq c$.

Suppose $i_{2c} \geq c$. By (27) there exists an index $p \leq 2c$ such that $i_p = c$. Since A is nilpotent of class $\leq c$ we have

$$\gamma_{i_{q-1}+1} A = \gamma_{c+1} A = \{1\} \qquad \text{for all } q > p.$$

Hence $\{1\} = \gamma_{i_{q-1}+1} A \cap H \subseteq \gamma_{j_{q-1}+1} B$ yielding $j_q = j_{q-1} + 1$ for all $q > p$ by (24) and (26). Thus we have

$$j_{2c} = j_{p+(2c-p)} = j_p + (2c - p). \tag{28}$$

By (27) we have $i_p + j_p \geq p$ and thereby $j_p \geq p - c$ since $i_p = c$. By (28) we get $j_{2c} \geq (p - c) + (2c - p) = c$. Thus we have

$$i_{2c} \geq c \quad \text{and} \quad j_{2c} \geq c \tag{29}$$

yielding $A_{2c+1} = (\gamma_{i_{2c}} A \cap \gamma_{j_{2c}+1} B)\gamma_{i_{2c}+1} A = \{1\}$ and

$$B_{2c+1} = (\gamma_{i_{2c}+1} A \cap \gamma_{j_{2c}} B)\gamma_{j_{2c}+1} B = \{1\}$$

because A and B are nilpotent groups of class $\leq c$.

For $j_{2c} \geq c$ we analogously get $A_{2c+1} = \{1\} = B_{2c+1}$.

iii) For each j there exists an index m such that $\gamma_j A = A_m$: For $j = 1$ we have $A_1 = \gamma_1 A$ by (21). For $j \geq c + 1$ we have $\gamma_j A = \{1\} = A_{2c+1}$ by ii).

Suppose $2 \leq j \leq c$: Since $i_{k-1} + 1 \geq i_k \geq i_{k-1}$ for all $k \geq 2$ by (27) and $i_{2c} \geq c$ by (29), there exists an index $m > 1$ such that $i_m = j$ and $i_m = i_{m-1} + 1$. Because $i_m = i_{m-1} + 1$ we have $\gamma_{j_{m-1}+1} B \cap H \subseteq \gamma_{i_{m-1}+1} A \cap H$ by (25) or (26). Hence

$$A_m = (\gamma_{i_{m-1}} A \cap \gamma_{j_{m-1}+1} B)\gamma_{i_{m-1}+1} A = \gamma_{i_{m-1}+1} A = \gamma_{i_m} A = \gamma_j A.$$

For each j there exists an index n such that $\gamma_j B = B_n$: follows analogously.

By *i)*, *ii)* and *iii)* we now have that $(A_k)_{k\in N}$ and $(B_k)_{k\in N}$ are refinements of length $\leq 2c$ of the lower central series of A and B respectively.

3) In the last step we show by induction on k that $A_k \cap H = B_k \cap H$ for all $k \geq 1$.

$k = 1$: By definition we have $A_1 \cap H = A \cap H = H = B \cap H = B_1 \cap H$.

$k = 2$: By definition we have

$$A_2 \cap H = (\gamma_1 A \cap \gamma_2 B)\gamma_2 A \cap H = (\gamma_2 B \cap H)(\gamma_2 A \cap H)$$

and

$$B_2 \cap H = (\gamma_2 A \cap \gamma_1 B)\gamma_2 B \cap H = (\gamma_2 A \cap H)(\gamma_2 B \cap H).$$

Hence $A_2 \cap H = B_2 \cap H$.

$k > 2$: Since $A *_H B$ is non-twisted we have by lemma 4

$$\gamma_{i_{k-1}+1} A \cap H \subseteq \gamma_{j_{k-1}+1} B \text{ or } \gamma_{j_{k-1}+1} B \cap H \subseteq \gamma_{i_{k-1}+1} A.$$

Suppose $\gamma_{i_{k-1}+1} A \cap H \subseteq \gamma_{j_{k-1}+1} B$: By (22) we have

$$A_k \cap H = (\gamma_{i_{k-1}} A \cap \gamma_{j_{k-1}+1} B)(\gamma_{i_{k-1}+1} A \cap H) \subseteq \gamma_{j_{k-1}+1} B \cap H \subseteq B_k \cap H$$

and by (23) we have $B_k \cap H = \gamma_{j_{k-1}+1} B \cap H$. To prove $A_k \cap H = B_k \cap H$ it therefore suffices to show that

$$\gamma_{j_{k-1}+1} B \cap H \subseteq A_k.$$

By the induction hypothesis we have $A_{k-1} \cap H = B_{k-1} \cap H$. Thus we get by (22) and (23)

$$(\gamma_{i_{k-2}} A \cap \gamma_{j_{k-2}+1} B)\gamma_{i_{k-2}+1} A \cap H = (\gamma_{i_{k-2}+1} A \cap \gamma_{j_{k-2}} B)\gamma_{j_{k-2}+1} B \cap H. \qquad (30)$$

By (24), (25) or (26) we either have $i_{k-1} = i_{k-2}$ or $i_{k-1} = i_{k-2} + 1$.

Suppose $i_{k-1} = i_{k-2}$: With $j_{k-1} \geq j_{k-2}$ and (30) we have

$$\gamma_{j_{k-1}+1} B \cap H \subseteq \gamma_{j_{k-2}+1} B \cap H \subseteq (\gamma_{i_{k-1}} A \cap \gamma_{j_{k-2}+1} B)\gamma_{i_{k-1}+1} A \subseteq \gamma_{i_{k-1}} A.$$

Thus we get by (22)

$$\gamma_{j_{k-1}+1} B \cap H \subseteq \gamma_{i_{k-1}} A \cap \gamma_{j_{k-1}+1} B \subseteq A_k.$$

Suppose $i_{k-1} = i_{k-2} + 1$: By (25) and (26) we have

$$\gamma_{j_{k-2}+1} B \cap H \subseteq \gamma_{i_{k-2}+1} A = \gamma_{i_{k-1}} A.$$

Since $j_{k-1} \geq j_{k-2}$ we get by (22)

$$\gamma_{j_{k-1}+1} B \cap H \subseteq \gamma_{i_{k-1}} A \cap \gamma_{j_{k-1}+1} B \subseteq A_k.$$

The case $\gamma_{j_{k-1}+1} B \cap H \subseteq \gamma_{i_{k-1}+1} A$ follows analogously. □

6.3 Additivity

In the following lemma 6 we show that if a refinement $(A_k)_{k \in N}$ of the lower central series of a nilpotent group contains a sufficient number of copies of each term of the lower central series then

$$[A_r, A_s] \subseteq A_{r+s} \qquad \text{for all } r \text{ and } s.$$

To navigate in $(A_k)_{k \in N}$ we need the auxiliary functions $\tau^{\min}, \tau^{\max} : N \to N$ which we define as follows: Let $\tau^{\min}(0) = 0$ and

$$\tau^{\min}(j) = \min\{r \in N \mid A_r = \gamma_j A\} \qquad (31)$$

$$\tau^{\max}(j) = \max\{r \in N \mid A_r = \gamma_j A \text{ and } r \leq l\} \qquad (32)$$

where $l = \min\{r \in N \mid A_r = \{1\}\}$ and j and r are positive integers. Thus $\tau^{\min}(j)$ is the index of the first and $\tau^{\max}(j)$ is the index of the last term in $(A_k)_{k \in N}$ which is equal to $\gamma_j A$.

Lemma 6 *Let A be a nilpotent group of class c and $(A_k)_{k \in N}$ a refinement of the lower central series of A. Suppose*

$$\tau^{\min}(j) - \tau^{\min}(j-1) \leq \tau^{\max}(j) - \tau^{\min}(j) + 1$$

holds for all positive integers $j \leq c$. Then

$$[A_r, A_s] \subseteq A_{r+s}$$

for all positive integers r and s.

Proof: Let r and s be positive integers. We may assume, without loss of generality, that $r \geq s$. With

$$\sigma(r) = \max\{j \in N \mid A_r \subseteq \gamma_j A \text{ and } j \leq c+1\}$$

we have

$$[A_r, A_s] \subseteq [\gamma_{\sigma(r)}A, \gamma_{\sigma(s)}A] \subseteq \gamma_{\sigma(r)+\sigma(s)}A = A_{\tau^{\max}(\sigma(r)+\sigma(s))}.$$

Suppose $\sigma(r) + \sigma(s) > c$. Since A is nilpotent of class $\leq c$ we have $\gamma_{\sigma(r)+\sigma(s)}A = \{1\}$ and therefore $[A_r, A_s] = \{1\} \subseteq A_{r+s}$. In order to prove lemma 6 it therefore suffices to show that

$$\tau^{\max}(\sigma(r) + \sigma(s)) \geq r + s$$

for all positive integers r and s with $\sigma(r) + \sigma(s) \leq c$.

Let j_1, j_2 and k be positive integers such that $j_1 + k \leq j_2 + k \leq c$. We first show that

$$\tau^{\min}(j_2) - \tau^{\min}(j_1) \leq \tau^{\min}(j_2 + k) - \tau^{\min}(j_1 + k) \tag{33}$$

holds. Since $\tau^{\max}(i) < \tau^{\min}(i+1)$ for all $i \leq c$ we have by the hypothesis

$$\tau^{\min}(i) - \tau^{\min}(i-1) \leq \tau^{\min}(i+1) - \tau^{\min}(i) \leq \tau^{\min}(i+k) - \tau^{\min}(i+k-1)$$

for all positive integers $k \leq c + 1 - i$. Hence

$$\begin{aligned}
\tau^{\min}(j_2) - \tau^{\min}(j_1) &\leq \sum_{i=j_1+1}^{j_2} \tau^{\min}(i) - \tau^{\min}(i-1) \\
&\leq \sum_{i=j_1+1}^{j_2} \tau^{\min}(i+k) - \tau^{\min}(i+k-1) \\
&\leq \tau^{\min}(j_2+k) - \tau^{\min}(j_1+k).
\end{aligned}$$

Thus inequality (33) holds for all k such that $j_1 + k \leq j_2 + k \leq c$.

Let r be a positive integer such that $\sigma(r) \leq c$. By the definition of σ we have $A_r \neq \{1\}$. Thus $A_{\tau^{\min}(\sigma(r)+1)} = \gamma_{\sigma(r)+1}A \subset A_r \subseteq \gamma_{\sigma(r)}A$ yielding $r \leq \tau^{\min}(\sigma(r)+1) - 1$.

Let r, s be positive integers such that $\sigma(r) + \sigma(s) \leq c$: By (33) and $\tau^{\min}(1) = 1$ we now have

$$\begin{aligned}
r + s &\leq \tau^{\min}(\sigma(r)+1) - 1 + \tau^{\min}(\sigma(s)+1) - 1 \\
&\leq \tau^{\min}(\sigma(r)+1) - 1 + \tau^{\min}(\sigma(r)+\sigma(s)) - \tau^{\min}(\sigma(r)) \\
&= \tau^{\min}(\sigma(r)+1) - \tau^{\min}(\sigma(r)) + \tau^{\min}(\sigma(r)+\sigma(s)) - 1.
\end{aligned}$$

Again by (33) and the hypothesis we get

$$\begin{aligned}
r + s &\leq \tau^{\min}(\sigma(r)+\sigma(s)) - \tau^{\min}(\sigma(r)+\sigma(s)-1) + \tau^{\min}(\sigma(r)+\sigma(s)) - 1 \\
&\leq \tau^{\max}(\sigma(r)+\sigma(s)).
\end{aligned}$$

Hence lemma 6 holds. □

6.4 Refining a Central Series

Let $G = G_1 *_H G_2$ be an amalgamation of nilpotent groups of class $\leq c$ and $(C_{q,k})_{k\in N}$ for $q = 1, 2$ a refinement of the lower central series of G_q such that $C_{1,k} \cap H = C_{2,k} \cap H$ for all k, c.f. section 6.2. Define $\tau_q^{\min}$ and $\tau_q^{\max}$ for $(C_{q,k})_{k\in N}$ as in section 6.3. Let

$$\mu_q = \max\{j \leq c+1 \mid \tau_q^{\min}(i) - \tau_q^{\min}(i-1) \leq \tau_q^{\max}(i) - \tau_q^{\min}(i) + 1 \text{ for all } i < j\}. \quad (34)$$

Thus if $\mu_q \leq c$ then μ_q is the index of the first term of the lower central series of G_q for which $(C_{q,k})_{k\in N}$ does not contain enough copies to satisfy the condition in lemma 6. We may assume, without loss of generality that $\tau_1^{\min}(\mu_1) \leq \tau_2^{\min}(\mu_2)$. Suppose $\tau_1^{\min}(\mu_1) \leq c$. Hence $(C_{1,k})_{k\in N}$ does not satisfy the condition of lemma 6. We refine in lemma 7 and lemma 8 the central series $(C_{q,k})_{k\in N}$ by inserting the required number of copies of $C_{1,\tau_1^{\min}(\mu_1)} = \gamma_{\mu_1} G_1$ after $C_{1,\tau_1^{\min}(\mu_1)}$ and the same number of copies of $C_{2,\tau_1^{\min}(\mu_1)}$ after $C_{2,\tau_1^{\min}(\mu_1)}$. Thereby we get refinements $(\tilde{C}_{q,k})_{k\in N}$ of $(C_{q,k})_{k\in N}$ such that $\tilde{\mu}_1 > \mu_1$ while preserving $\tilde{C}_{1,k} \cap H = \tilde{C}_{2,k} \cap H$. In proposition 2 of section 6.5 we will iteratively refine the central series until $\tilde{\mu}_q = c+1$ for $q = 1, 2$. Thereby the resulting central series will satsify the condition in lemma 6. Let l_q be the length of $(C_{q,k})_{k\in N}$ plus 1. In lemma 7 we construct the refinements for the case $\tau_1^{\min}(\mu_1) < l_q$ for $q = 1, 2$ and in lemma 8 for the case $\tau_1^{\min}(\mu_1) = l_1 < l_2$.

Lemma 7 *Let $G = G_1 *_H G_2$ with G_q for $q = 1, 2$ a non-trivial nilpotent group of class $\leq c$ and H a subgroup of G_q. Let $(C_{q,k})_{k\in N}$ be a refinement of the lower central series of G_q such that $C_{1,k} \cap H = C_{2,k} \cap H$ for all k. Define $\tau_q^{\min}$, $\tau_q^{\max}$ and μ_q for $(C_{q,k})_{k\in N}$ as in (31), (32) and (34) respectively. Let $t = \min\{\tau_1^{\min}(\mu_1), \tau_2^{\min}(\mu_2)\}$, $l_q = \min\{k \in N \mid C_{q,k} = \{1\}\}$ and $l = \max\{l_1, l_2\}$. Suppose*

$$0 < l_q - t < 2c \quad \text{and} \quad l \leq 2^{2c-(l-t)}c.$$

There exists a refinement $(\tilde{C}_{q,k})_{k\in N}$ for $q = 1, 2$ of $(C_{q,k})_{k\in N}$ such that $\tilde{C}_{1,k} \cap H = \tilde{C}_{2,k} \cap H$ for all k and

$$0 \leq \tilde{l}_q - \tilde{t} < l_q - t < 2c \quad \text{and} \quad \tilde{l} \leq 2^{2c-(\tilde{l}-\tilde{t})}c$$

with $\tilde{t}$, $\tilde{l}_q$ defined for $(\tilde{C}_{q,k})_{k\in N}$ as t, l for $(C_{q,k})_{k\in N}$ above.

Proof: We may assume, without loss of generality, that $t = \tau_1^{\min}(\mu_1) \leq \tau_2^{\min}(\mu_2)$. Let

$$s = \max\{2\tau_q^{\min}(\mu_q) - \tau_q^{\min}(\mu_q - 1) - \tau_q^{\max}(\mu_q) - 1 \mid q = 1, 2\}.$$

Since $\tau_1^{\min}(\mu_1) = t < l_1$ we have $\mu_1 \leq c$ and therefore

$$\tau_1^{\min}(\mu_1) - \tau_1^{\min}(\mu_1 - 1) > \tau_1^{\max}(\mu_1) - \tau_1^{\min}(\mu_1) + 1.$$

by the definition of μ_1. Hence $s > 0$. We construct the new refinement $(\tilde{C}_{q,k})_{k\in N}$ by inserting s copies of $C_{q,t}$ after $C_{q,t}$:

$$\begin{aligned} \tilde{C}_{q,k} &= C_{q,k} && \text{for} \quad k = 1, \ldots, t \\ \tilde{C}_{q,k} &= C_{q,t} && \text{for} \quad k = t+1, \ldots, t+s \\ \tilde{C}_{q,k} &= C_{q,k-s} && \text{for} \quad k = t+s+1, \ldots \end{aligned}$$

Thus $\tilde{C}_{1,k} \cap H = \tilde{C}_{2,k} \cap H$ since $C_{1,k} \cap H = C_{2,k} \cap H$.

We define $\tilde{\tau}_q^{\min}$, $\tilde{\tau}_q^{\max}$, $\tilde{\mu}_q$, $\tilde{t}$ and $\tilde{l}_q$ for $(\tilde{C}_{q,k})_{k\in N}$ as above.

First, we show $\tilde{\tau}_1^{\min}(\tilde{\mu}_1) > t+s$: Since $\tilde{C}_{q,k} = C_{q,k}$ for $k \leq t$ we have $\tilde{\mu}_1 \geq \mu_1$. By construction we have

$$\tilde{\tau}_1^{\min}(i) = \tau_1^{\min}(i) \text{ for } i \leq \mu_1 \quad \text{and} \quad \tilde{\tau}_1^{\max}(\mu_1) = \tau_1^{\max}(\mu_1) + s.$$

Thus we get

$$\begin{aligned}
\tilde{\tau}_1^{\min}(\mu_1) - \tilde{\tau}_1^{\min}(\mu_1 - 1) &= \tau_1^{\min}(\mu_1) - \tau_1^{\min}(\mu_1 - 1) \\
&\leq s - \tau_1^{\min}(\mu_1) + \tau_1^{\max}(\mu_1) + 1 \\
&\leq \tilde{\tau}_1^{\max}(\mu_1) - \tilde{\tau}_1^{\min}(\mu_1) + 1.
\end{aligned}$$

Hence $\tilde{\mu}_1 > \mu_1$. Since $l_1 > t$ we have $C_{1,t} \neq \{1\}$ and thereby $\tilde{\tau}_1^{\min}(\mu_1+1) > \tilde{\tau}_1^{\max}(\mu_1)$. Thus

$$\tilde{\tau}_1^{\min}(\tilde{\mu}_1) > \tilde{\tau}_1^{\max}(\mu_1) = \tau_1^{\max}(\mu_1) + s \geq \tau_1^{\min}(\mu_1) + s = t + s.$$

Next, we show $\tilde{\tau}_2^{\min}(\tilde{\mu}_2) > t+s$:

Suppose $\tau_2^{\min}(\mu_2) = t$: By the definition of s we get $\tilde{\tau}_2^{\min}(\tilde{\mu}_2) > t+s$ as above.

Suppose $\tau_2^{\min}(\mu_2) > t$: By construction we have $\tilde{\tau}_2^{\min}(\tilde{\mu}_2) > t$ yielding $\tilde{\tau}_2^{\min}(\tilde{\mu}_2) > t+s$ since $\tilde{C}_{2,t} = \tilde{C}_{2,t+s}$.

We now have $\tilde{\tau}_q^{\min}(\tilde{\mu}_q) > t+s$ for $q = 1,2$ yielding

$$\tilde{t} = \min\{\tilde{\tau}_1^{\min}(\tilde{\mu}_1), \tilde{\tau}_2^{\min}(\tilde{\mu}_2)\} > t+s.$$

Since $t < l_1$ we have $C_{1,t} \neq \{1\}$ and thereby $\tilde{l}_q = l_q + s$. Hence we get

$$0 \leq \tilde{l}_q - \tilde{t} = l_q + s - \tilde{t} < l_q - t < 2c \quad \text{and} \quad \tilde{l}_q - \tilde{t} < l_q - t \leq l - t.$$

Together with

$$s \leq \max\{2\tau_q^{\min}(\mu_q) - \tau_q^{\max}(\mu_q) \,|\, q = 1,2\} \leq \max\{\tau_1^{\min}(\mu_1), \tau_2^{\min}(\mu_2)\} \leq l$$

and the hypothesis we eventually have

$$\tilde{l} = l + s \leq 2l \leq 2 \cdot 2^{2c-(l-t)}c \leq 2^{2c-(\tilde{l}-\tilde{t})}c.$$

□

Lemma 8 *Let $G = G_1 *_H G_2$ with G_q for $q = 1,2$ a non-trivial nilpotent group of class $\leq c$ and H a subgroup of G_q. Let $(C_{q,k})_{k\in N}$ be a refinement of the lower central series of G_q such that $C_{1,k} \cap H = C_{2,k} \cap H$ for all k. Define $\tau_q^{\min}$, μ_q and l_q for $(C_{q,k})_{k\in N}$ as in lemma 7. Suppose $l_1 = \tau_1^{\min}(\mu_1) \leq \tau_2^{\min}(\mu_2)$, $0 < l_2 - \tau_2^{\min}(\mu_2) < 2c$ and $l_2 \leq 2^{2c-(l_2-\tau_2^{\min}(\mu_2))}c$.*

There exists a refinement $(\tilde{C}_{2,k})_{k\in N}$ of $(C_{2,k})_{k\in N}$ such that

$$C_{1,k} \cap H = \tilde{C}_{2,k} \cap H \text{ for all } k, \quad 0 \leq \tilde{l}_2 - \tilde{\tau}_2^{\min}(\tilde{\mu}_2) < l_2 - \tau_2^{\min}(\mu_2) < 2c$$

and $\tilde{l}_2 \leq 2^{2c-(\tilde{l}_2-\tilde{\tau}_2^{\min}(\tilde{\mu}_2))}c$ with $\tilde{\tau}_2^{\min}$, $\tilde{\mu}_2$ and $\tilde{l}_2$ defined for $(\tilde{C}_{2,k})_{k\in N}$ as in lemma 7.

Proof: Let $s = 2\tau_2^{\min}(\mu_2) - \tau_2^{\min}(\mu_2 - 1) - \tau_2^{\max}(\mu_2) - 1$. Since $\tau_2^{\min}(\mu_2) < l_2$ we have $\mu_2 \leq c$ and therefore $s > 0$ by the definition of μ_2. We construct the refinement $(\tilde{C}_{2,k})_{k\in N}$ of $(C_{2,k})_{k\in N}$ by inserting s copies of $C_{2,t}$ after $C_{2,t}$ as in lemma 7. Hence $C_{1,k} \cap H = C_{2,k} \cap H = \tilde{C}_{2,k} \cap H$ for all $k \leq \tau_2^{\min}(\mu_2)$. Since $l_1 = \tau_1^{\min}(\mu_1) \leq \tau_2^{\min}(\mu_2)$ we have for all $k \geq \tau_2^{\min}(\mu_2)$

$$C_{1,k} \cap H = \{1\} \subseteq \tilde{C}_{2,k} \cap H \subseteq \tilde{C}_{2,\tau_2^{\min}(\mu_2)} \cap H = C_{2,\tau_2^{\min}(\mu_2)} \cap H = \{1\}.$$

Hence $C_{1,k} \cap H = \tilde{C}_{2,k} \cap H$ for all k.

We define $\tilde{\tau}_2^{\min}$, $\tilde{\tau}_2^{\max}$, $\tilde{\mu}_2$ and $\tilde{l}_2$ for $(\tilde{C}_{2,k})_{k\in N}$ as above.

We show $\tilde{\tau}_2^{\min}(\tilde{\mu}_2) > \tau_2^{\min}(\mu_2) + s$: Since $\tilde{C}_{2,k} = C_{2,k}$ for $k \leq \tau_2^{\min}(\mu_2)$ we have $\tilde{\mu}_2 \geq \mu_2$. By construction we have

$$\tilde{\tau}_2^{\min}(i) = \tau_2^{\min}(i) \text{ for } i \leq \mu_2 \quad \text{and} \quad \tilde{\tau}_2^{\max}(\mu_2) = \tau_1^{\max}(\mu_2) + s.$$

Thus we get

$$\begin{aligned} \tilde{\tau}_2^{\min}(\mu_2) - \tilde{\tau}_2^{\min}(\mu_2 - 1) &= \tau_2^{\min}(\mu_2) - \tau_2^{\min}(\mu_2 - 1) \\ &\leq s - \tau_2^{\min}(\mu_2) + \tau_2^{\max}(\mu_2) + 1 \\ &\leq \tilde{\tau}_2^{\max}(\mu_2) - \tilde{\tau}_1^{\min}(\mu_2) + 1. \end{aligned}$$

Hence $\tilde{\mu}_2 > \mu_2$ and therefore

Since $l_2 > \tau_2^{\min}(\mu_2)$ we have $C_{1,\tau_2^{\min}(\mu_2)} \neq \{1\}$ and thereby $\tilde{\tau}_2^{\min}(\mu_2 + 1) > \tilde{\tau}_2^{\max}(\mu_2)$. Thus

$$\tilde{\tau}_2^{\min}(\tilde{\mu}_2) > \tilde{\tau}_2^{\max}(\mu_2) = \tau_2^{\max}(\mu_2) + s \geq \tau_2^{\min}(\mu_2) + s.$$

Also by $C_{2,\tau_2^{\min}(\mu_2)} \neq \{1\}$ we have $\tilde{l}_2 = l_2 + s$. Hence we get $0 \leq \tilde{l}_2 - \tilde{\tau}_2^{\min}(\tilde{\mu}_2) = l_2 + s - \tilde{\tau}_2^{\min}(\tilde{\mu}_2) < l_2 - \tau_2^{\min}(\mu_2) < 2c$ and $\tilde{l}_2 - \tilde{\tau}_2^{\min}(\tilde{\mu}_2) < l_2$. Together with $s \leq \tau_2^{\min}(\mu_2) < l_2$ and the hypothesis we eventually have

$$\tilde{l}_2 = l_2 + s \leq 2l_2 \leq 2 \cdot 2^{2c-(l_2 - \tau_2^{\min}(\mu_2))}c \leq 2^{2c-(\tilde{l}_2 - \tilde{\tau}_2^{\min}(\tilde{\mu}_2))}c.$$

□

6.5 Main Result

Let $G = G_1 *_H G_2$ be a non-twisted amalgamation of nilpotent groups. Combining the results of section 6.2, section 6.3 and section 6.4 we construct in proposition 2 a central series $(C_{q,k})_{k\in N}$ for G_q for $q = 1, 2$ such that

$$C_{1,k} \cap H = C_{2,k} \cap H \quad \text{and} \quad [C_{q,r}, C_{q,s}] \subseteq C_{q,r+s}.$$

By proposition 1 of section 4 we then get in theorem 3 a rewriting process ρ from G to H such that δ_ρ and Φ_ρ are bounded above by a polynomial. Since H is also finitely generated and nilpotent, Φ_H is bounded above by a polynomial, c.f. remark 1. Thereby we get in theorem 4 that

$$\Phi_G(n) \preceq \Phi_\rho(n) + \Phi_H(\delta_\rho(n))$$

is bounded above by a polynomial, our main result.

Proposition 2 *Let $G = G_1 *_H G_2$ be a non-twisted amalgamation with G_q for $q = 1, 2$ a non-trivial nilpotent group of class $\leq c$ and H a subgroup of G_q. There exists a refinement $(C_{q,k})_{k\in N}$ of length $\leq 2^{2c}c$ of the lower central series of G_q such that*

$$C_{1,k} \cap H = C_{2,k} \cap H \quad \text{and} \quad [C_{q,r}, C_{q,s}] \subseteq C_{q,r+s}$$

for all positive integers k, r and s.

Proof: Since $G_1 *_H G_2$ is non-twisted there exists by lemma 5 a refinement $(A_{q,k})_{k\in N}$ of length $\leq 2c$ of the lower central series of G_q such that $A_{1,k} \cap H = A_{2,k} \cap H$ for all k. By iterated application of lemma 7 we get a further refinement $(B_{q,k})_{k\in N}$ such that $B_{1,k}\cap H = B_{2,k}\cap H$ for all k and

$$0 \leq l_{B,q} - t_B < 2c, \quad l_B \leq 2^{2c-(l_B-t_B)}c \quad \text{and} \quad l_{B,1} = t_B \text{ or } l_{B,2} = t_B. \qquad (35)$$

with $l_{B,q}$, l_B and t_B defined for $(B_{q,k})_{k\in N}$ as in lemma 7. We may assume, without loss of generality, that $l_{B,1} = t_B$. Let $C_{1,k} = B_{1,k}$ for all k. By iterated application of lemma 8 to $(B_{2,k})_{k\in N}$ we get a refinement $(C_{2,k})_{k\in N}$ of $(B_{2,k})_{k\in N}$ such that

$$C_{1,k}\cap H = C_{2,k}\cap H \text{ for all } k, \quad l_{C,q} = \tau^{\min}_{C,q}(\mu_{C,q}) \quad \text{and} \quad l_{C,q} \leq 2^{2c}c.$$

with $\tau^{\min}_{C,q}$, $\mu_{C,q}$,$l_{C,q}$ defined for $(C_{q,k})_{k\in N}$ as in lemma 7. Since $l_{C,q} = \tau^{\min}_{C,q}(\mu_{C,q})$ we have $\mu_{C,q} = c+1$ and therefore

$$\tau^{\min}_{C,q}(j) - \tau^{\min}_{q,q}(j-1) \leq \tau^{\max}(j) - \tau^{\min}(j) + 1$$

for all $j < \mu_{C,q} = c+1$ by the definition of $\mu_{C,q}$. Hence we get by lemma 6

$$[C_{q,r}, C_{q,s}] \subseteq C_{q,r+s} \quad \text{for all } r \text{ and } s.$$

□

Theorem 3 *Let $G = G_1 *_H G_2$ be a* non-twisted amalgamation *with G_q for $q = 1,2$ a finitely generated* nilpotent *group of class c and H a subgroup of G_q. There exists a rewriting process ρ from G to H such that*

$$\delta_\rho(n) \preceq n^{2^{(2c)}c} \quad \textit{and} \quad \Phi_\rho(n) \preceq n^{2^{(2c+1)}c+1}.$$

Proof: We may assume, without loss of generality, that G_q for $q = 1,2$ is not trivial. By proposition 2 there exists a central series $(N_{q,k})_{k\in N}$ of G_q of length $\leq 2^{2c}c$ such that $N_{1,k}\cap H = N_{2,k}\cap H$ and $[N_{q,r}, N_{q,s}] \subseteq N_{q,r+s}$ for $q = 1,2$ and all positive integers k, r and s. By proposition 1 there exists therefore a rewriting process ρ from G to H such that

$$\delta_\rho(n) \preceq n^{2^{(2c)}c} \quad \text{and} \quad \Phi_\rho(n) \preceq n^{2^{(2c+1)}c+1}.$$

□

Theorem 4 *Let G be a* non-twisted *amalgamation of finitely generated* nilpotent *groups of class c. Then*

$$\Phi_G(n) \preceq n^{2^{(2c+1)}c^2}.$$

Proof: For $c \leq 1$ we have $\Phi_G(n) \preceq n^2$, c.f. [BGSS91, Hid97]. Let $c \geq 2$ and let $G = G_1 *_H G_2$. By theorem 3 there exists a rewriting process ρ from G to H such that

$$\delta_\rho(n) \preceq n^{2^{(2c)}c} \quad \text{and} \quad \Phi_\rho(n) \preceq n^{2^{(2c+1)}c+1}.$$

Since $H \subseteq G_q$ and G_q is a finitely generated nilpotent group of class $\leq c$ the subgroup H is also finitely generated and nilpotent of class $\leq c$. Hence $\Phi_H(n) \preceq n^{2c}$, c.f. remark 1. We now get

$$\Phi_G(n) \preceq \Phi_\rho(n) + \Phi_H(\delta_\rho(n)) \preceq n^{2^{(2c+1)}c+1} + n^{2^{(2c+1)}c^2} \preceq n^{2^{(2c+1)}c^2}$$

since $c \geq 2$. □

7 Amalgamation along Abelian Subgroups

We show in theorem 5 that an amalgamation of finitely generated nilpotent groups along a suitable abelian subgroup satisfies a polynomial isoperimetric inequality. However, there exist amalgamations along abelian subgroups having an exponential isoperimetric function and we give in theorem 7 an example. Besides being a twisted amalgamation, i.e. not non-twisted, along an abelian subgroup, the example's subgroup is also isolated and normal.

Theorem 5 *Let $G = G_1 *_H G_2$ be an amalgamation of finitely generated* nilpotent *groups. Suppose*

$$H \subseteq \gamma_j G_q \quad and \quad H \cap \gamma_{j+1} G_q = \{1\}$$

for some positive integer j and $q = 1$ or $q = 2$. Hence H is abelian. Then G is non-twisted *and thereby satisfies a* polynomial isoperimetric inequality*.*

Proof: We may assume, without loss of generality, that $H \subseteq \gamma_j G_1$ and $H \cap \gamma_{j+1} G_1 = \{1\}$ for some j. Let i and k be some positive integers such that $\gamma_i G_1 \cap H \not\subseteq \gamma_k G_2$. Hence $\gamma_i G_1 \cap H \neq \{1\}$ and therefore $i \geq j$. Thus

$$\gamma_k G_2 \cap H \subseteq H = \gamma_j G_1 \cap H \subseteq \gamma_i G_1 \cap H.$$

Thus G is a non-twisted amalgamation and therefore satisfies a polynomial isoperimetric inequality by theorem 4. □

We give in theorem 7 an example of a twisted amalgam of finitely generated nilpotent groups along an abelian, isolated and normal subgroup having an exponential isoperimetric function. To this end we need the following result due to M. Bridson:

Theorem 6 *[Bri95, Main Theorem] The Dehn function for any finite presentation of a semidirect product of the form $A \lhd_\Psi F$, with A a finitely generated abelian group and F a finitely generated free group, is $\simeq$ equivalent to either a polynomial or an exponential function.*

The action of F on A via Ψ induces an action on A modulo its torsion subgroup and hence a representation $\sigma : F \to Gl_m(Z)$, where $m = rk_Z A$. The Dehn function of $A \lhd_\Psi F$ is polynomial iff there exists a subgroup of finite index $\tilde{F} \subseteq F$ such that $\sigma(\tilde{F}) \subseteq Gl_m(Z)$ consists entirely of unipotent elements; the degree of the polynomial is then d, where

$$d - 2 = \Lambda(im\Psi) := \max\{r | N_1 N_2 \cdots N_r \neq 0 \; for\ some\ I + N_i \in \sigma(\tilde{F})\}.$$

In particular

$$d \leq rk_Z A + 1.$$

A subgroup H of a group G is called *isolated* if and only if $g^n \in H$ implies $g \in H$ for all $g \in G$ and all non-zero integers n.

Theorem 7 *Let G_i for $i = 1, 2$ be the free nilpotent group of class 2 and rank 2. There exists a twisted amalgamation $G = G_1 *_H G_2$ with H* abelian, isolated *and* normal *such that*

$$\Phi_G(n) \cong 2^n.$$

Proof: Let $G_1 = \langle a, b, c \,|\, [b, a]c^{-1}, [b, c], [a, c]\rangle$, the free nilpotent group of class 2 and rank 2 and H_1 the subgroup generated by b and c. Let $G_2 = \langle a', b', c' \,|\, [b', a']c'^{-1}, [b', c'], [a', c']\rangle$ and $H_2 = \langle b', c'\rangle \subset G_2$. Note that $H_i \cong Z^2$ for $i = 1, 2$ is an abelian, isolated and normal subgroup of G_i. Let $\varphi : H_1 \longrightarrow H_2$ be the isomorphism given by $b \mapsto c'$ and $c \mapsto b'$. Let $G = G_1 *_H G_2$ with $H \cong H_1 \cong H_2$ and let φ be the amalgamation isomorphism.

Since $\gamma_2 G_1 \cap H = \langle c\rangle = \langle b'\rangle \not\subseteq \gamma_2 G_2 \cap H$ and

$$\gamma_2 G_2 \cap H = \langle c'\rangle = \langle b\rangle \not\subseteq \gamma_2 G_1 \cap H$$

the amalgamation G is twisted.

We use the same notation as in theorem 6: Let $F =< a, a' >\subset G$, a free subgroup of rank 2. We now have $G = FH$, H is normal in G and $H \cap F = \{\varepsilon\}$. Let $\Psi : F \to AutH$ with $\Psi(v)(h) = v^{-1}hv$ for $v \in F$ and $h \in H$, hence $G = H \lhd_\Psi F$. For $b^r c^s$ an arbitrary element of H we have

$$\Psi(a)(b^r c^s) = a^{-1} b^r c^s a =_G a^{-1} b^r a c^s =_G b^r c^{r+s}$$

and analogously $\Psi(a')(b^r c^s) =_G b^{r+s} c^s$. Let σ be the representation of theorem 6. Hence $\sigma(a) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $\sigma(a') = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Let $\tilde{F} \subseteq F$ be a subgroup of finite index and $\alpha = a' a \in F$. Since $\tilde{F}$ is of finite index there exist $n, m > 0$ such that α^n and α^{n+m} are in the same coset of $\tilde{F}$, yielding $\alpha^m \in \tilde{F}$. Because $\sigma(\alpha) = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ has an eigenvalue > 1, $\sigma(\alpha^m)$ also has an eigenvalue > 1. Therefore $\sigma(\alpha^m)$ is not unipotent. Hence we get $\Phi_G(n) \cong 2^n$ by theorem 6. □

Note that the amalgamated subgroup in theorem 7 is exponentially distorted, i.e. $\delta_{H,G}(n) \cong 2^n$, cf. [Hid97].

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