# Polynomial Time Approximation of Dense Weighted Instances of MAX-CUT

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TR-97-059

December 1997

**Abstract**

We give the first polynomial time approximability characterization of dense weighted instances of MAX-CUT, and some other dense weighted $\mathcal{NP}$-hard problems in terms of their empirical weight distributions. This gives also the first almost sharp characterization of inapproximability of unweighted 0,1 MAX-BISECTION instances in terms of their density parameter only.

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# 1 Introduction

Significant results concerning polynomial time approximation schemes (PTAS's) for "dense" instances of several $\mathcal{NP}$-hard problems such as MAX-CUT, MAX-k-SAT, BISECTION, DENSE-k-SUBGRAPH, and others have been obtained recently in Arora, Karger and Karpinski [AKK95], Fernandez de la Vega [FV96], Arora, Frieze and Kaplan [AFK96], Frieze and Kannan [AK97]. Still more recently, the approximability of dense instances of $\mathcal{NP}$-hard problems has been investigated from the point of view of the query complexity. Goldreich, Goldwasser and Ron [GGR96] show that a constant size sample is sufficient to test wether a graph has a cut of a certain size. Frieze and Kannan [AK97], obtain quick approximations for all dense MAX-SNP problems. Recall that a PTAS for a given optimization problem is a family $(A_\epsilon)$ of algorithms indexed by a parameter $\epsilon \in (0, \infty)$ where each algorithm runs in polynomial time and, for each $\epsilon$, the algorithm $A_\epsilon$ has approximation ratio $1 - \epsilon$ (or $1 + \epsilon$ for a minimization problem). In most cases, the instances are graphs, and a dense graph is defined as a graph with $\Theta(n^2)$ edges where $n$ is the number of vertices. (In some cases, the algorithms apply only to graphs with minimum degree $\Theta(n)$.) Some of the problems considered in the papers mentioned above, such as MAX-CUT, are MAX-SNP-hard, and thus, if $\mathcal{P} \neq \mathcal{NP}$, have no PTASs when the set of instances is not restricted. Let us also mention that the PTASs in [FV96], [AK97] and [GGR96] are *efficient* in the sense of Cesati and Trevisan [CS97].

The natural instances of optimization problems (see the definitions given in [GJ79]) involve weights while the results mentioned above deal mainly with the 0,1 case. The purpose of this paper is to examine how these results can be extended to the weighted case. We want to define a concept of density for the weighted case which ensures that our algorithms, possibly with minor modifications, work in the corresponding *dense* classes of instances and such that the *non-dense* classes are not approximable under a standard intractability assumption. For the sake of simplicity, we concentrate here on MAX-CUT. In fact, for technical reasons, we start by considering MAX-BISECTION, which is MAX-CUT restricted to cuts with equal sides. (MAX-BISECTION is also called MAX-50/50-CUT or MAX-EQUI-CUT.) Our results extend easily to other MAX-SNP-hard problems such as MAX-2SAT or MAXIMUM ACYCLIC SUBGRAPH. We remark in passing that the methods of [AKK95] and [FV96] give a PTAS for MAX-BISECTION.

We note that weight problems have been briefly considered in [GGR96] and [AK97]. In both papers, the authors evaluate the increase of the computation time of their algorithms when one allows weights belonging to some fixed interval $[0, a]$ instead of 0,1 weights. Weight problems are also considered in a recent paper [TR97].

The plan of this paper is as follows. In sections 2 and 3, we define and characterize our dense classes of weighted instances via classes of distribution functions (d.f.'s for short) of the weights. They clearly grasp the intuitive, and standard notions of dense instances of combinatorial optimization problems. In section 5, we prove that MAX-BISECTION and MAX-CUT both have PTASs in any dense class of weighted instances according to our definition. In section 6 we prove that both MAX-BISECTION and MAX-CUT are MAX-SNP hard on any fixed non-dense set of weighted instances satisfying a certain mild additional condition. The last section contains a summary and open problems.

## 2 Definition of a Dense family

In as much as density requirements come in, any given instance is a set of non-negative real numbers (the weights) or rather a multi set. Let us associate to this instance the empirical distribution function of the weights:

$$F(x) = \frac{2}{n(n-1)} \sum_{x_i \le x} m_i, \quad x \in \mathbb{R}^+$$

where $m_i$ denotes the multiplicity of the weight $x_i$ in the instance and $n$ is the number of vertices.

*We define our density classes in terms of families of weight distribution functions.* More precisely:

(i) To each d.f. $F$ with support in $\mathbb{R}^+$, we associate the set $I_F$ of all weighted graphs whose empirical weight distribution coincides with $F$.

(ii) To each set $\mathcal{F}$ of d.f.'s we associate the set of instances

$$I_{\mathcal{F}} = \cup_{F \in \mathcal{F}} \, I_F$$

Thus, we shall define below *dense* sets of d.f.'s having in mind the sets of instances to which they correspond according to (i) and (ii).

Clearly, our d.f.'s need have finite discrete support and rational individual probabilities. (We don't dwell here about the nature of the values in the support. For definitness, let us say that they are also rational.) We call such d.f.'s *representable*. Conversely, the set of instances corresponding to a representable d.f. $F$ with individual probabilities having smallest common denominator $d$, say, is given by

$$I_F = \cup_{\{n : 2d | n(n-1)\}} \mathcal{G}_n$$

where $\mathcal{G}_n$ is the set of weighted graphs on $n$ vertices whose empirical weight distribution coincides with $F$. Notice that $I_F$ is infinite for any representable $F$. For convenience, when representativity is not essential, we state in various occasions our theorems in terms of arbitrary d.f.'s. (not necessarily having finite or even discrete range). Moreover, we often assume in our proofs that our d.f.'s are continuous (i.e. we assume that the corresponding random variables have densities). Whenever we make this assumption, it is justified by the fact that any d.f. can be approximated arbitrarily closely by a continuous one.

We can assume that the mean of the weights in each instance is equal to 1, since, when we divide all the weights by their mean, say $m$, we also divide the values of the objective function by $m$ so that the approximation ratios are unaffected. (We assume that the weights are not all 0.) Nevertheless, we shall not always impose this condition in the definition of our dense families. Since our definitions are invariant under scaling we shall assume in the proofs that the d.f.'s have expectation 1. (Here and all along the paper, we speak with some abuse of language, of the expectation of a d.f. $F$ meaning the expectation of a random variable with d.f. $F$.)

We can now state our definition of a dense family of d.f.'s.

**Definition 1 (Dense families of d.f.'s)** *Let $\mathcal{F} = (F_j)_{j\in\mathcal{J}}$ be a family of integrable d.f.'s with supports contained in $\mathbb{R}^+$ and each with mean 1. Let $\mu_j$ denote the expectation of $F_j$. For each $j \in \mathcal{J}$ and each $k \in \mathbb{N}$, define*

$$M_{j,k} = \frac{1}{k\mu_j}\sum_{i=1}^{k} X_{j,i}$$

*where the $X_{j,i}$ are independent r.v.'s each with d.f. $F_j$.*

*We say that the family $\mathcal{F}$ is dense if and only if, for each $j \in \mathcal{J}$, the sequence $(M_{j,k})_{k=1,2,..}$ converges in probability to 1, and moreover, this convergence is uniform for $j \in \mathcal{J}$.*

In other words, $\mathcal{F}$ is dense iff there exists a function $n_\epsilon = n(\epsilon) : (0,1] \to \mathbb{N}$ such that the inequalities

$$\mathsf{Pr}\ [|M_{j,k} - 1| \leq \epsilon] \geq (1-\epsilon),\ \ k \geq n_\epsilon \tag{1}$$

hold for each $\epsilon$ and simultaneously for all $j$, with an $n_\epsilon$ which depends only on $\epsilon$ (and not on $j$).

**Definition 2** *A family $\mathcal{F}$ of d.f.'s which is not dense is called a non-dense family*

REMARK. Our characterization of dense families of d.f.'s does not require that the d.f.'s be representable. Of course, our approximability or inapproximability theorems will in fact be concerned with *representable* families.

An instant reflexion shows that condition (i) brings just what we want. If $\mathcal{F}$ is dense in the sense of definition 1, then we can estimate the mean of each $F \in \mathcal{F}$ with any desired relative accuracy by picking a sample whose size $n_\epsilon$ does not depend on $F$. In the next section we identify some natural dense families of d.f.'s.

## 3 Some Dense Families of d.f.'s

Recall the law of large numbers: If $X$ has a finite mean $EX$, then the means of the partial sums of a sequence of independent random variables each distributed as $X$ converges in probability to $EX$. This implies immediately the next proposition.

**Proposition 1** *Any singleton (and any finite set of integrable d.f.'s) with support in $\mathbb{R}^+$ is a dense family*

The following assertion can easily be checked.

**Proposition 2** *The family of all integrable d.f.'s is not dense*

Our next dense family defines precisely, when restricted to 0,1 instances, the usual density classes. In the unweighted case (and after scaling), dense means "not too dispersed". We can thus define dense families based on some dispersion measure. The most common of these is the variance, and this leads to the following class of dense families.

**Proposition 3** *For each $s \geq 0$ the family*

$$\mathcal{F}_s = \left\{ F_X : \frac{\text{Var}X}{(EX)^2} \leq s \right\}$$

*is dense.*

PROOF The proof is straightforward by using Chebyshev's inequality. □

The last example can be generalized as follows.

**Proposition 4** *For each pair $(r, C)$ where $r \in (1, +\infty)$ and $C \in \mathbb{R}^+$, the family*

$$\mathcal{F} = \{F_X : \frac{1}{(EX)^r} \int_0^\infty x^r dF_X(x) \leq C\} \tag{2}$$

*is dense.*

PROOF Fix $r \in (1, +\infty]$ and $C \in \mathbb{R}^+$ and let $\mathcal{F}$ be the corresponding family of d.f.'s defined in proposition 4 where we can suppose $EX = 1$ for every $X$. The inequality (2) gives immediately, for any $t \in \mathbb{R}^+$,

$$1 - F(t) = \int_t^\infty dF(x) \leq Ct^{-r},$$

We have thus

$$t(1 - F(t)) \leq Ct^{1-r}$$

whose right hand side tends to 0 when $t \to \infty$, uniformly for $F \in \mathcal{F}$. Anticipating our characterization of the dense families (see Theorem 1 in the next section), we deduce that $\mathcal{F}$ is dense. □

## 4 A Characterization of the Dense Families

The following theorem characterizes the dense families. Once again, this theorem, alike Definition 1, is stated in terms of arbitrary (not necessarily representable) d.f.'s.

**Theorem 1** *Let $\mathcal{F} = (F_j)_{j \in \mathcal{J}}$ be a family of non-negative integrable d.f.'s and assume all expectations equal to 1.*

*The family $\mathcal{F}$ is dense in the sense of Definition 1 if and only if one of the following conditions (i) and (ii) holds:*

*(i) For each $j$ and each $x \in \mathbb{R}^+$, define $\tau_j(x) = x(1 - F_j(x))$. There is a function $\tau_o(x)$ tending to 0 as $x \to \infty$ and such that the inequalities*

$$\tau_j(x) \leq \tau_o(x) \tag{3}$$

*hold for each pair $(j, x)$.*

*(ii) For each $j$ and each $x \in \mathbb{R}^+$, define*

$$s_j(x) = \int_x^\infty y dF_j(y). \tag{4}$$

*There is a function $s_o(x)$ tending to 0 as $x \to \infty$ and such that the inequalities*

$$s_j(x) \leq s_o(x) \tag{5}$$

*hold for each pair $(j, x)$.*

We shall also use occasionally the following characterization of the *non-dense* families.

*Assume that $\mathcal{F}$ is not dense. Then there is an $\eta > 0$ such that, for any arbitrary large $y \in \mathbb{R}^+$, there is an $F \in \mathcal{F}$ with*

$$y(1 - F_j(y)) \geq \eta. \tag{6}$$

To see this, note that the contrary would state:

$\forall \eta > 0 \ \exists y(\eta) \in \mathbb{R}^+$ s.t. $y(\eta)(1 - F_j(y(\eta))) < \eta$
for every $F \in \mathcal{F}$

Then, putting $y_k = y(2^{-k})$, we could define an $\tau_o$ for $\mathcal{F}$ by $\tau_o(x) = 2^-k$ for $y_k \leq x < y_{k+1}$, which contradicts the assumption that $\mathcal{F}$ is not dense.

PROOF (of Theorem 1). Notice that if we take $s_o = \tau_o$, then ii) is more stringent than i). Thus, it suffices to show that (ii) is necessary and (i) sufficient.

The fact that condition (i) implies that the family $\mathcal{F}$ is dense, can be established easily by adapting the proof of the law of large numbers in order to get an effective bound on the sample size. Actually, we will adapt a proof of Feller (see [Fe]) that he uses to show the convergence of the means of sums of independent r.v.'s to a not necessary constant specified function. The speed of convergence is governed by the function $\tau$. Let us write

$$S_n = X_1 + \dots + X_n$$

where the $X_i$ are independent with the common d.f. $F$ with expectation $\mu$. Let us define new r.v.'s $X_i'$ by truncation at level $n$:

$$X_i' = X_i \text{ if } X_i \leq n, \qquad X_i' = 0 \text{ if } X_i > n.$$

Put

$$S_n' = X_1' + \dots + X_n', \qquad m_n' = E(S_n') = nE(X_1').$$

Then,

$$\begin{aligned} P[|S_n - m_n'| > t] &\leq P[|S_n' - m_n'| > t] \\ &\quad + P[S_n \neq S_n']. \end{aligned}$$

Putting $t = n\epsilon$ and applying Chebyshev's inequality to the first term on the right, we get

$$\Pr\,[|S_n - m_n'| > t] \leq \frac{1}{n^2\epsilon^2}E(X_1'^2) + nP[X_1 > n] \tag{7}$$

Put

$$\sigma(t) = \int_0^t x^2 dF(x).$$

Then, an integration by parts gives

$$\begin{aligned} \sigma(n) &= -n\tau(n) + 2\int_0^n x\tau(x)dx \\ &\leq 2\int_0^n x\tau(x)dx. \end{aligned}$$

(Recall that $\tau(x) = x(1 - F(x))$.) We have thus, for each $n$,

$$\begin{aligned} \Pr\left[\left|\frac{S_n}{n} - EX_1'\right| \geq \epsilon\right] &\leq \frac{2}{n^2\epsilon^2}\int_0^n x\tau(x)dx \\ &\quad + \tau(n) \end{aligned}$$

Since $EX_1'$ tends to $EX_1$ uniformly for $F \in \mathcal{F}$ as $n \to \infty$, this implies

$$\Pr\left[\left|\frac{S_n}{n} - EX_1\right| \geq 2\epsilon\right] \leq \frac{2\int_0^n x\tau(x)dx}{n^2\epsilon^2} + \tau(n) \tag{8}$$

for sufficiently large $n$. Clearly, the right side tends to 0 again uniformly whenever $\tau(t) \leq \tau_o(t)$ with a $\tau_o(t) \to 0$. This concludes the proof of sufficiency of condition (i).

The proof of the necessity of condition (ii) again mimics standard arguments. We omit it in this extended abstract. □

# 5 A PTAS for Dense Weighted Instances of MAX-BISECTION and MAX-CUT

In [AKK95] and [FV96] the following Theorem was proved.

**Theorem 2.** *0,1 dense MAX-CUT does have a PTAS.*

The following more general result can be proved in a similar way.

**Theorem 3.** *Assume that the set of instances $I$ has been standardized (i.e. the mean of the weights is 1) and moreover assume that the weights are bounded above by an absolute constant. Then MAX-CUT and MAX-BISECTION on $I$ both have PTASs.*

The crux of the methods of [AKK95] and [FV96] relies on so-called sampling lemmas which work when the dispersion of the weights is of comparable magnitude to that of their means. This is guaranteed by the assumptions of Theorem 3.

The following Theorem will be easily deduced from Theorem 3.

**Theorem 4.** *Let the family of representable d.f.'s $\mathcal{F}$ be dense (i.e. each $F \in \mathcal{F}$ has a finite support and rational probabilities and, moreover, $\mathcal{F}$ satisfies to the conditions of Theorem 1). Then* MAX-CUT, *and* MAX-BISECTION *both have PTASs when restricted to the instances corresponding to $\mathcal{F}$.*

PROOF We first need some notation. Given an underlying vertex set $V = V_n$ of size $n$ and any subset $S \subseteq V$ we denote by $\delta(S)$ $(= \delta(V - S))$ the set of unordered pairs $uv$ of vertices with $u \in S,\ v \in V - S$. Thus $\delta(S)$ is the cut defined by $S$ in the complete graph with vertex set $V_n$.
For any instance $I$ and any subset $S$ of the corresponding graph, we denote by $val(I, S)$ the value of the cut defined by $S$:

$$val(I, S) := \Sigma_{e \in \delta(S)} w(e)$$

Here $w(e)$ is the weight of the edge $e$. If the instance is a graph, we write more simply $val(G, S)$ for $val(I, S)$. Hence we have

$$val(G, S) = |\delta(S) \cap E(G)|$$

where $E(G)$ denotes the edge set of $G$. Turning to the proof of Theorem 4, let $\mathcal{F}$ be dense, fix an $\epsilon > 0$ and let $m_o$ be the minimum real number such that the inequality

$$s_o(m_o) \leq \frac{\epsilon}{2}$$

is satisfied. Here $s_o(.)$ is the function corresponding to $\mathcal{F}$ in condition (i) of Theorem 1. Now let $I$ be an instance whose weight distribution coincides with some $F \in \mathcal{F}$). In order to approximate the maximum cut of $I$ within $1-\epsilon$ we can proceed as follows.

• We replace by 0 all the weights exceeding $m_o$. Let $I'$ denote the new entry.

• Since $I'$ has bounded weights after standardisation, we can according to Theorem 3, find in polynomial time a cut $\delta(S)$ whose value $val(I', S)$ approximates that of a maximum cut of $(I')$ within $1-\epsilon/2$, say.

Now, to see that $\delta(S)$ solves MAX-CUT within $1-\epsilon$ on the original instance $I$, observe that the total weight annihilated when going from $I$ to $I'$ does not exceed $\binom{n}{2}.\epsilon/2$. Thus, if $\mathrm{Opt}(I)$ is the maximum value of a cut of $I$, we have certainly

$$\begin{aligned} \frac{val(I',S)}{\mathrm{Opt}(I)} &\geq \frac{val(I',S)}{\mathrm{Opt}(I')} . \frac{\mathrm{Opt}(I')}{\mathrm{Opt}(I)} . \\ &\geq (1-\epsilon/2)^2 \geq 1-\epsilon \end{aligned}$$

where we have used in the last derivation the inequality $\mathrm{Opt}(I) \geq \frac{1}{2}\binom{n}{2}$. This concludes the proof for MAX-CUT. The proof for MAX-BISECTION is exactly the same. □

# 6 Hardness of MAX-BISECTION on a Non-Dense Set of Instances

The strict converse of Theorem 4 which would state that MAX-BISECTION and MAX-CUT are MAX-SNP-hard on any non-dense set $\mathcal{F}$ does not hold. To see this, let us recall first the precise result of [FV96] or [GGR96].

**Theorem 5** *For any fixed $d > 0$ and relative accuracy requirement $\epsilon$, there is an algorithm which solves MAX-CUT on instances of density at least $d$ in time at most*

$$C_1 n 2^{\frac{C_2}{d\epsilon^2}} \tag{9}$$

*where $C_1$ and $C_2$ are absolute constants*

Now let $\mathcal{F} = (F_i)_{i=1,2,\ldots}$ be a non-dense family of d.f's where $F_i$ corresponds to the 0,1 instances with density $d_i$, say, and the sequence $(d_i)$ tends to 0. (We always assume that the sequence $(d_i)$ decreases.). Let $\frac{N_i}{D_i} = d_i$ be the shortest fraction expressing $d_i$. Then $D_i$ divides $\binom{n_i}{2}$ where $n_i$ the smallest order of a graph where $F_i$ can be represented. Thus we certainly have $n_i \geq \sqrt{D_i}$. Assume $D_i \geq 2^{\frac{\lambda}{d_i}}$ for some fixed $\lambda > 0$ and all $i$. Then, the order $n$ of any graph on which $F_i$ is representable satisfies the inequality $n \geq 2^{\frac{\lambda}{2d_i}}$. Thus, according to 9, the time complexity $T(n)$ for computing MAX-CUT within $1-\epsilon$ on such a graph satisfies

$$T(n) \leq C_1 n 2^{\frac{C_2}{d_i \epsilon^2}} \leq C_1 n^{1+\frac{2C_2}{\lambda \epsilon^2}},$$

i.e. we have a PTAS for $\mathcal{F}$ with exponent $1+\frac{2C_2}{\lambda\epsilon^2}$.

We thus need an upper bound for the denominators of the $d_i$'s to obtain an inapproximability result in the 0,1 case and we will assume that the $D_i$'s are bounded above by a polynomial function of the inverse of the density. We shall use a similar condition in the general weighted case (see Theorem 7). Besides these small denominators conditions, the proofs of the inapproximability results that we present require another condition which, in the 0,1 case, says roughly speaking, that the sequence of densities $(d_i)$ does not decrease too fast (albeit it may decrease as fast as a double exponential). Let us now state these results.

**Theorem 6** (MAX-SNP-hardness of MAX-BISECTION in the non-dense 0,1 case) *Assume that the sequence of rational densities $(d_i)$ tends to 0 and, moreover, that it satisfies to the inequalities*

$$d_{i+1} \geq d_i^h, \; i = 1, 2, \ldots \tag{10}$$

*where $h$ is a positive constant. Assume moreover that the denominators $D_i$ of the $d_i$ satisfy*

$$D_i \leq p(d_i^{-1}) \tag{11}$$

*where $p(.)$ is a fixed polynomial.*

*Then, MAX-BISECTION is MAX-SNP-hard on the set of 0,1 instances whose densities belong to $(d_i)$.*

**Theorem 7** (MAX-SNP-hardness of MAX-BISECTION in the non-dense weighted case) *Let $\mathcal{F} = (F_i)_{i=1,2,\ldots}$ be a non-dense family of representable d.f.'s and, for each $i$, let $D_i$ denote the smallest common denominator of the individual probabilities of the distribution $F_i$. Assume that there exists reals $\eta > 0$ and $h > 1$, and a sequence of numbers $(t_i)_{i=1,2,\ldots}$ tending to infinity, s.t. the following three conditions hold for all $i \in \mathbb{N}$:*

$$t_i(1 - F_i(t_i)) \geq \eta, \tag{12}$$

$$D_i \leq p(t_i) \tag{13}$$

*and*

$$t_{i+1} \leq t_i^h. \tag{14}$$

*Then, MAX-BISECTION is MAX-SNP-hard on the set of instances corresponding to $\mathcal{F}$.*

**Theorem 8** (MAX-SNP-hardness of MAX-CUT in the non-dense weighted case) *Let $\mathcal{F} = (F_i)_{i=1,2,...}$ be a non-dense family of representable d.f.'s and assume that $\mathcal{F}$ fulfills the conditions of Theorem 7. Then, MAX-CUT is MAX-SNP-hard on the set of instances corresponding to $\mathcal{F}$.*

Before turning to the proofs of these theorems, let us give some words of explanation about conditions (10) and (11) in Theorem 6. (Theorem 7 uses similar conditions (13) and (14).) In the set of 0,1 instances corresponding to non-dense family $\mathcal{F}$, the density $d$ achieves arbitrarily small values. Thus, from the fact that MAX-BISECTION (or MAX-CUT) is MAX-SNP hard on 0,1 instances with bounded degree (see Papadimitriou and Yannakakis [PY91]), there is apparently nothing to prove to obtain Theorem 6 where it not because of the apparently innocuous fact that, for each involved $d$, we are not required to solve MAX-BISECTION on all possible instances of density $d$, but only for a set of sizes which may be quite thin. Conditions (10) and (14) are introduced to keep these sizes under control.

## 6.1 Hardness of MAX-BISECTION on a Non-Dense Set of Unweighted Instances

We need several lemmas.

Let $\mathcal{G}(n, d)$ denote the set of graphs with $n$ vertices and average degree $d$.

**Lemma 3** *Let an integer $h$ and a graph $G$ of order $n$ be given. Let $H$ denote the join of $G$ with an independent set of size $h$ (i.e. we make $h$ replicas of each vertex of $G$ and each edge of $G$ gives a complete bipartite graph between the two corresponding sets of replicas). Assume $1 \leq h \leq p(n)$ where $p(.)$ is a fixed polynomial. Then, the problems of approximating MAX-BISECTION in $H$ and in $G$ are mutually L-reducible one to the other.*

PROOF For each vertex $x \in V(G)$, its $h$ replicas are equivalent and thus we can assume that they all go to the same side of the cuts of $H$. But then the natural correspondence between the equi-cuts of $G$ and the cuts of $H$ with this property leaves the approximation ratios unchanged and the Lemma follows. □

**Lemma 4** *Let $\Delta$ be a sufficiently large real number and let $\Delta' > \Delta$. MAX-BISECTION is MAX-SNP-hard on any set of graphs*

$$\mathcal{H} = \cup_{n \in \mathbb{N}} \mathcal{G}(n, d_n)$$

*where the $d_n$'s satisfy $\Delta \leq d_n \leq \Delta'$ for each $n \in \mathbb{N}$.*

PROOF The result of Papadimitriou and Yannakakis [PY91] quoted in the previous section implies that MAX-BISECTION is MAX-SNP-hard on graphs whose average degree has a fixed upper bound, say $D$. We shall define an L-reduction from these graphs to graphs in $\mathcal{H}$ for a fixed sequence $(d_n)$. Let $G$ be a graph, with $n$ vertices and average degree $\delta \leq D$. Clearly, we can assume $\delta \geq 1$. Let the $d_n$'s satisfy to the condition of the Theorem and put $h = \lfloor \frac{d_n}{\delta} \rfloor$. Denote by $G'$ be the join of $G$ by an independent set of size $h$. Then, by adding less than $2d_n\delta^{-1}|E(G)|$ arbitrary edges to $G'$, we obtain a graph $G$" with average degree $d_n$. Using Lemma 3, one can easily verify that the mapping $G \to G$" provides (for sufficiently large $\Delta$) the desired L-reduction. □

**Lemma 5** *Assume that the sequence $(n_k)_{k=1,\ldots}$ satisfies for any sufficiently large $k$ to the inequality*

$$n_{k+1} \leq n_k^h \tag{15}$$

*where $h$ is a fixed number greater than 1. Then, MAX-BISECTION is MAX-SNP-hard when restricted to graphs whose vertex set sizes belong to $(n_k)$.*

PROOF Assume for a contradiction that for any $\epsilon \geq 0$, there exists an integer $k$ such that 0,1 MAX-BISECTION is $(1-\epsilon)$-approximable in time $n^k$ for vertex set sizes belonging to $(n_j)$ by some algorithm $A$. Set for each $n$, $m = m(n) = \min\{n_j : n_j \geq \frac{n}{\epsilon}\} = n_q$, say. We have $m \leq n^{h+1}$ for sufficiently large $n$. Let $\lambda = \lfloor \frac{m}{n} \rfloor$ and associate to each instance $I$ of size $n$ the join $J$ of $\lambda$ copies of $I$. Eventually add isolated vertices to obtain an instance $J'$ of order $m$. Clearly, an approximate solution of $J'$ is also an approximate solution of $J$ and by Lemma 3, we can deduce in polynomial time from an approximate solution of $J'$ an approximate solution of $I$ with the same approximation ratio. Thus the algorithm $A$ can be used with trivial modifications to approximate MAX-BISECTION for an instance of size $n$ in time $n^{kh}$. This contradicts the MAX-SNP-hardness of unweighted MAX-BISECTION. □

**Lemma 6** *Let $\Delta$ and $\Delta'$ be defined as in Lemma 4 and assume that the sequence $(n_i)_{i=1,2,\ldots}$ satisfies*

$$n_{i+1} \leq n_i^h$$

*for some real $h > 1$. Assume moreover that the sequence $(d_i)_{i=1,2,\ldots}$ satisfies $\Delta \leq d_i \leq \Delta'$ for each $i$. Then, MAX-BISECTION is MAX-SNP-hard on the set*

$$\cup_{i \in \mathbb{N}} \mathcal{G}(n_i, d_i)$$

PROOF Starting from Lemma 4, the proof is completely similar to that of Lemma 5 and is omitted. □

We are now well prepared for the proof of Theorem 6.

**Proof of Theorem 6**

We shall give an L-reduction to 0,1 MAX-BISECTION. Let the sequence of densities $\mathcal{D} = (d_i)$ satisfy to the conditions of the Theorem. Fix an arbitrary small $\epsilon > 0$ and define from $\mathcal{D}$ a new family $\mathcal{D}'$ where for each $i$, $d_i$ is replaced by a $\delta_i$ satisfying

$$(1-\epsilon)d_i \leq \delta_i \leq d_i$$

and having a shortest fractional expression, say $\delta_i = \frac{P_i}{Q_i}$, with $\lceil \frac{1}{\epsilon} \rceil \leq P_i \leq \lceil \frac{2}{\epsilon} \rceil$. Let us show first that MAX-BISECTION is hard to approximate on $\mathcal{D}'$. Because of Lemma 5 we need only an infinite sequence of sizes $(n_k)$ such that, for each $k$, the average degrees $\bar{d}_k$ of the graphs on $n_k$ vertices and with density $d_k$ in our family of instances belong to some fixed interval $[\Delta, \Delta']$ with $\Delta$ sufficiently large.For a graph on $n_k$ vertices with density $\delta_k$ we have

$$\bar{d}_k = (n_k - 1)\delta_k = (n_k - 1).\frac{P_k}{Q_k}$$

Thus, if we choose $n_k = \Delta Q_k + 1$, we get $\bar{d}_k = \Delta P_k$ implying $\Delta \leq \bar{d}_k \leq \Delta'$ as desired. It remains to observe that (10) implies the inequality

$$\delta_{i+1} \geq \delta_i^{h+1}$$

for all sufficiently large $i$.

In order to conclude the promised reduction, we shall show that MAX-BISECTION on $\mathcal{D}'$ L-reduces to MAX-BISECTION on $\mathcal{D}$.

Such a reduction can be obtained as follows: For each sufficiently large $i$, pick the smallest $j$ such that $\frac{d_j}{\delta_i} \geq \frac{2}{\epsilon}$ Make the join of an instance of density $\delta_i$ with an independent set of size $h_i = \lfloor \frac{d_j}{\delta_i} \rfloor$, and add the necessary number of edges to obtain an instance with density $d_j$. Using Lemma 5, it is easy to see that this transformation is an L-reduction. □

The following Lemma asserts broadly speaking that putting random weights with mean 1 on the edges of a (not too sparse) graph $G$ does not change significantly the maximum value of a cut of $G$.

**Lemma 7 (Averaging Lemma)** *Let $(G_n)_{n=1,\ldots}$ be a sequence of graphs where $G_n$ has $n$ vertices and $m = m(n)$ edges and $n = o(m)$. Assume that for each $n$ the edges of $G_n$ are given random non-negative weights picked from a fixed distribution $F$ with mean 1. Let $G'_n$ denote this weighted graph.*
*The quantity*

$$\frac{1}{m} \max_S |val(G'_n, S) - val(G_n, S)|,$$

*where $S$ ranges over all subsets of $V(G)$, tends to 0 in probability when $n \to \infty$.*

PROOF Is given in the Appendix.

## 6.2 End of the Proof of Theorem 7

Our strategy for obtaining an hardness result for the general (weighted) case of MAX-BISECTION is to reduce it to the 0,1 case.
We are going to prove the following Theorem

**Theorem 9** *Assume that $\mathcal{F}$ satisfies to the conditions of Theorem 7 with parameters $\eta$ and $h$. Then, approximating MAX-BISECTION on $\mathcal{F}$ L-reduces to approximating MAX-BISECTION on a non-dense set of 0,1 instances.*

PROOF Let the sequences $(t_i)$ and $(D_i)$ satisfy to the conditions of Theorem 7 and let $\mu_i = 1 - F(t_i)$. We have $t_i \geq \frac{\eta}{\mu_i}$. Thus (13) implies

$$D_i \leq q(\mu_i^{-1}) \tag{16}$$

where $q$ is another polynomial. Also, it is not hard to show that (14) implies the existence of a subsequence (j(i)) of the natural integers with

$$\mu_{j(i+1)} \geq (\mu_{j(i)})^{h+1}.$$

We can thus assume by renaming that we have

$$\mu_{i+1} \geq \mu_i^{h+1} \tag{17}$$

for every $i \in \mathbb{N}$.

For a fixed $i$, set $F \equiv F_i,\ t = t_i$ and define

$$\alpha = \alpha_i = \frac{1}{1 - F(t)} \int_t^\infty s dF(s)$$

and

$$\beta = \beta_i = \frac{1}{F(t)} \int_0^t s dF(s).$$

For $n = 2\lambda D_i$, $\lambda \in \mathbb{N}$, set $m = \mu_i \binom{n}{2}$, ($m$ is an integer because $F_i$ is representable on a graph with $n$ vertices), and use $k$ to index the $\binom{\binom{n}{2}}{m}$ distinct subgraphs $G_k$ of $K_n$ having $m$ edges.
We define for each $k$ a partial instance $J_k$ by giving to the edges of $G_k$ random weights empirically distributed according to the d.f.

$$G(s) = \frac{F(s) - F(t)}{1 - F(t)},\ s \geq t.$$

We define also a partial instance $L_k$ by putting on the edges in the set $K_n \backslash G_k$ random weights on $[0, t]$ empirically distributed according to the d.f. $H(s) = F(s)/F(t),\ s \leq t$. We denote by $I_k$ the instance obtained by sticking together $J_k$ and $L_k$. Clearly, by the choice of $G(.)$ and $H(.)$, the empirical distribution of the weights in $I_k$ coincides with $F$.

Assume that one can find in polynomial time a bisection $\delta(S_o)$ with value $val(I_k, S_o) \geq (1-\epsilon)Opt(I_k)$. Let $I'_k$ denote the instance obtained by replacing the weights in $J_k$ (resp. in $L_k$) by their mean $\alpha$, (resp. $\beta$).
By applying the averaging Lemma separately to $J_k$ and $L_k$, we see that the maximum value of a bisection in $I'_k$ does not differ from $val(I'_k, S_o)$ by more than a $1 - o(1)$ factor.
Let now $I"_k$ denote the instance obtained from $I'_k$ by substracting $\beta$ from each weight. (Thus, $I"_k$ has weights all equal to $\alpha - \beta$ on the edges of $G_k$ and zero weights elsewhere.) For any bisection $\delta(S)$ we have

$$val(I'_k, S) = val(I"_k, S) + \frac{\beta n(n-1)}{4} \ .$$

Note that we have $\alpha \geq \eta$ and since we have

$$\alpha(1 - F(t)) + \beta F(t) = 1,$$

while $F(t) = F_i(t_i)$ tends to 1 with $i$, we deduce that $\beta$ has an upper bound strictly smaller than 1. Since the maximum value of a bisection of $G$ is at least $n^2/4$, this implies for $S = S_o$ that the the ratio

$$\frac{val(I"_k, S_o)}{val(I'_k, S_o)}$$

is bounded below by a strictly positive constant so that $\delta(S_o)$ is also an approximate solution for MAX-BISECTION on $I"_k$. Thus, approximating MAX-BISECTION on $\mathcal{F}$ enables us to approximate 0,1 MAX-BISECTION on the graphs with densities $(\mu_i)$ and orders $D_i$, under conditions (16) and (17.) This clearly contradicts Theorem 7. □

## 7 Proof of Theorem 8

In order to prove Theorem 8, we consider only for each $F \in \mathcal{F}$ and each $n$, the graphs $G_k$ whose vertex set is contained in the first half of the vertex set $V_n$ of the instance $I_k$. Let us denote by $J_{\mathcal{F}}$ this restricted set of instances. ($G_k$, $I_k$ and $I'_k$ are as defined in the end of the previous section.) We are going to show that, for each instance $I_k \in J_{\mathcal{F}}$, we can easily deduce a nearly optimal bisection from any nearly optimal cut. Now, reasoning as in the end of the previous section, one can show that approximating MAX-BISECTION on $J_{\mathcal{F}}$ amounts to approximate MAX-BISECTION on a non-dense set of 0,1 instances and Theorem 8 follows.

Using the sampling Lemma as in the proof of Theorem 7, it suffice to prove the claim with $I_k$ replaced by $I'_k$. For $S \subseteq V_n$, we have

$$\begin{aligned} val(I'_k, S) &= \alpha|\delta(S) \cap G_k| + \beta(|\delta(S)| \\ &- |\delta(S) \cap G_k|) \\ val(I'_k, S) &= (\alpha - \beta)|\delta(S) \cap G_k| \\ &+ \beta|\delta(S)| \end{aligned} \tag{18}$$

Because of our assumption on $G_k$, $|\delta(S) \cap G_k|$ can be maximised with $|S| = \frac{n}{2}$, and this choice maximizes also $|\delta(S)|$. We have thus

$$Opt(I'_k) = (\alpha - \beta)Opt(G_k) + \frac{1}{4}\beta n^2. \tag{19}$$

where $Opt(I'_k)$ denotes the common maximum value of MAX-CUT and MAX-BISECTION on $I'_k$. Let us write $|\delta(S)| = |S|(n - |S|) = \frac{1}{4}n^2 - m$. Put $|S| = \frac{n}{2} - m$. (18) gives also

$$val(I'_k, S) \leq (\alpha - \beta)Opt(G_k) + \beta(\frac{n^2}{4} - m^2)$$

Thus, comparing with (19) and using the bound $Opt(I'_k) \leq \frac{1}{2}\binom{n}{2}$, we see that we have

$$m^2 \leq \frac{\epsilon^2 n^2}{2\beta}$$

for any $S$ such that $val(I'_k, S) \geq Opt(I'_k)(1 - \epsilon^2)$.

Let $\delta(S')$ denote a bisection obtained from the cut $\delta(S)$ by moving from the biggest to the smallest side of $\delta(S)$ $m$ vertices incident only $\beta$-edges. There are two cases (i) and (ii):

(i) If $\beta \geq \epsilon$, then the inequality

$$val(I'_k, S') \geq val(I'_k, S) - m^2$$

gives

$$\begin{aligned} val(I'_k, S') &\geq Opt(I'_k)(1 - \epsilon^2) - \frac{\epsilon n^2}{2} \\ &\geq Opt(I'_k)(1 - 2\epsilon - \epsilon^2), \end{aligned}$$

(using again the inequality $Opt(I'_k) \geq \frac{n^2}{4}$) i.e. $\delta(S')$ is a bisection of $I'_k$ with approximation ratio $1 - \epsilon'$, where $\epsilon' = 2\epsilon + \epsilon^2$.

(i) If $\beta < \epsilon$, then we have

$$val(I'_k, S') \geq val(I'_k, S) - \frac{\epsilon n^2}{2}$$

which leads to the same conclusion as case (i).

□

## 8 Summary and Conclusions

With the aim of separating as sharply as possible the approximable from the inapproximable families of weighted instances of MAX-CUT, we have introduced a notion of dense families of instances or, more precisely, a notion of dense families of weight distributions. We have shown that the corresponding families of instances have the (intended) approximability property for MAX-CUT.

In the other direction, we have shown inapproximability only when the densities in the set of instances do not decrease too fast, and we believe that this condition is not necessary. This is our first question.

A second question is: Does our density definition capture the approximability of all MAX-SNP-hard problems in the weighted case? We know by [AKK95] that all these problems are approximable in the dense *unweighted* case.

## Acknowledgement

We thank Sanjeev Arora, Martin Dyer and Alan Frieze for helpful discussions.

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## Appendix.

**Proof of the Averaging Lemma (Lemma 7).** We prove that, for every $\epsilon > 0$, for sufficiently large $n$,

$$\begin{aligned} \mathsf{Pr}[\max_S |val(G'_n, S) &- val(G_n, S)| \leq \epsilon m] \\ &\geq 1 - \epsilon \ . \end{aligned}$$

We prove the inequality

$$\begin{aligned} \mathsf{Pr}[\max_S (val(G'_n, S) &- val(G_n, S)) \leq \epsilon m] \\ &\geq 1 - \frac{\epsilon}{2} \ . \end{aligned} \tag{20}$$

We first get rid of the extreme values of $F$. Define $\theta = \theta(\epsilon)$ by

$$\int_\theta^\infty s dF(s) = \frac{\epsilon^2}{16}. \tag{21}$$

Then the expectation of the total weight of the edges with weights $\geq \theta$ is equal to $\frac{m\epsilon^2}{16}$. Markov inequality inequality implies that, with probability at least $1 - \frac{\epsilon}{4}$, this weight does not exceed $\frac{m\epsilon}{4}$.

Let us fix a cut $\delta(S)$ with $val(G_n, S) = |\delta(S) \cup E(G_n)| = q$, say, with $1 \leq q \leq m$. The value $val(G'_n, S)$ of this cut for $G'_n$ is the sum of $q$ independent r.v.'s with the common distribution $F$. Call $F_1$ the distribution obtained from $F$ by cutting the values $\geq \theta$: $F_1(s) = F(s)/(1 - F(\theta))$, $s \leq \theta$. Let us denote by $E_1$ the expectation of $F_1$. By what has just been proved, we have with probability at least $1 - \frac{\epsilon}{4}$, and simultaneously for all $S$,

$$val(G'_n, S) \leq \Sigma(S) + \frac{m\epsilon}{4} \tag{22}$$

where $\Sigma(S)$ is the sum of $\lambda m$ independent r.v.'s with the common distribution $F_1$. Using the fact that these r.v.'s are bounded above by $\theta$ and the martingale inequality of Azuma (see [ASE92], Appendix A3) , we obtain

$$\begin{aligned} \mathsf{Pr}[(\Sigma(S) - qE_1) &\geq \frac{\epsilon m}{4}] \\ &\leq \exp\left(-\frac{\epsilon^2 m^2}{32\theta^2 q}\right) \\ &\leq \exp\left(-\frac{\epsilon^2 m}{32\theta^2}\right). \end{aligned}$$

We have $val(G_n, S) = q$ and $E_1 \leq 1$. The preceding inequality gives, with (22),

$$\begin{aligned} \mathsf{Pr}[val(G'_n, S) &- val(G_n, S) \geq \frac{\epsilon m}{2}] \\ &\leq \exp\left(-\frac{\epsilon^2 m}{32\theta^2}\right). \end{aligned}$$

Since the total number of cuts is bounded above by $2^n$, we obtain

$$\begin{aligned} \mathsf{Pr}[val(G'_n, S) &- val(G_n, S) \geq \frac{\epsilon m}{2}, \ \forall S] \\ &\leq 2^n \exp\left(-\frac{\epsilon^2 m}{32\theta^2}\right) \\ &\leq \frac{\epsilon}{4} \end{aligned}$$

for sufficiently large $n$ by our assumption on $m = m(n)$. Since inequality (22) is valid with probability at least $1 - \frac{\epsilon}{4}$, inequality (20) is proved. The proof of the reverse inequality is similar and is omitted.

□