

Computing Irreducible Representations of Supersolvable Groups over Small Finite Fields

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Abstract

We present an algorithm to compute a full set of irreducible representations of a supersolvable group G over a finite field K , $\text{char}K \nmid |G|$, which is not assumed to be a splitting field of G . The main subroutines of our algorithm are a modification of the algorithm of Baum and Clausen [1] to obtain information on algebraically conjugate representations, and an effective version of Speiser's generalization of Hilbert's Theorem 90 stating that $H^1(\text{Gal}(L/K), \text{GL}(n, L))$ vanishes for all $n \geq 1$.

1 Introduction and Main Results

Recently Baum and Clausen [1] published an efficient algorithm for computing the absolutely irreducible representations of a supersolvable group G given in pc-presentation. The matrix representations their algorithm computes are adapted to a chief series $\mathcal{T} := (G = G_n > G_{n-1} > \cdots > G_0 = \{1\})$, i.e., any such representation D satisfies the following conditions: (1) the restriction $D \downarrow G_j$ of D to G_j is *equal* to a direct sum of irreducible matrix representations of G_j , and (2) equivalent irreducible constituents of $D \downarrow G_j$ are *equal*. The algorithm traverses the chief series \mathcal{T} bottom-up and constructs in each step j among other data a complete set of nonequivalent absolutely irreducible representations of G_j . These representations are almost unique: if L is a field containing a primitive e th root of unity, e being the exponent of G , and D and Δ are two equivalent irreducible \mathcal{T} -adapted representations of LG of degree d , say, then the *intertwining space*

$$\text{Int}(D, \Delta) := \{X \in L^{d \times d} \mid \forall g \in G : XD(g) = \Delta(g)X\}$$

is generated over L by a *monomial* matrix (see [2, Theorem 7.4]).

Now let K be a finite field, G be a supersolvable group such that $\text{char}K \nmid |G|$, \mathcal{T} be a chief series of G , and L be a finite extension of K which contains a primitive e th root of unity. The Galois group $\text{Gal}(L/K)$ acts on the irreducible matrix representations of LG in a straight forward manner. In Section 2 we shall modify the algorithm of Baum and Clausen by collecting at each step information about the $\text{Gal}(L/K)$ -orbits of the representations constructed. We then employ the information obtained at level n to compute realizations of direct sums of these representations over the field K . By a realization of a matrix representation D of LG over K we mean a matrix $T \in \text{GL}(d, L)$, d being the degree of D , such that $T^{-1}D(g)T$ has entries in K for all $g \in G$. Not every representation has a realization over K . Even more, if K is a prime field, χ denotes the character of D , and $K(\chi) := K(\chi(g) \mid g \in G)$ its character field, then D cannot have a realization over a proper subfield of $K(\chi)$. The question whether an absolutely irreducible representation D of G has a realization over $K(\chi)$ is hard to answer in general, i.e., for arbitrary K and arbitrary G . (This amounts to the question whether the Schur-index of the character of D equals 1, see [3, Kapitel V, §14].) It is however well-known that for finite fields and arbitrary finite groups the question has an affirmative answer [3, Kapitel V, Satz 14.10].

In theory we thus know that any irreducible matrix representation D of LG has a realization over its character field. How can we compute such a realization? Let M be a subfield of L of index ℓ , and β be the Frobenius automorphism of L/M . If D is an irreducible representation of LG of degree d , then so is D^β , where $D^\beta(g) := D(g)^\beta$ for all $g \in G$. If M is the character field of D , then D is equivalent to D^β , hence $\text{Int}(D, D^\beta)$ is generated by an invertible matrix S . A generalization of Hilbert's Theorem 90 due to Speiser [7] states that the first cohomology $H^1(\langle \beta \rangle, \text{GL}(d, L))$ is trivial. (This is a modern interpretation of Speiser's result; see also [6, Chapter X, §1].) Hence, for $S \in \text{GL}(d, L)$ there exists $T \in \text{GL}(d, L)$ such that $T^{-1}T^\beta = S$ if and only if the norm $S^{\beta^{\ell-1}} \cdots S^\beta S$ of S equals the $n \times n$ -identity matrix I_n . Such a matrix T will give the desired realization of D over its character field M . In our applications, S is a *monomial* matrix and this allows to compute T from S efficiently, see Section 3.

Now we are almost done. Namely, we may suppose that D is an absolutely irreducible

representation of G with character χ such that $D(g)$ has entries in $K(\chi)$. Let σ be the Frobenius automorphism of $K(\chi)/K$. Then the *trace* of D over K defined as $\text{Tr}_K(D) := D \oplus D^\sigma \oplus \cdots \oplus D^{\sigma^{m-1}}$, $m := [K(\chi):K]$, has character field equal to K and a realization of $\text{Tr}_K(D)$ over K can be computed easily, see Section 3. Furthermore, $\text{Tr}_K(D)$ is an irreducible KG -representation (since any of its irreducible constituents over K has to be invariant under σ); conversely, any irreducible KG -representation is the trace over K of some irreducible representation of LG . (For these and related facts see [4, Chapter VII, §1].) To obtain the irreducible representations of KG we first compute a set \mathcal{F}' of representatives of $\text{Gal}(L/K)$ -orbits of irreducible representations of LG , and for each such representation a realization of its trace over K . Starting from a pc-presentation of G and the chief series \mathcal{T} induced by that, the first two steps of our algorithm are as follows:

Step 1. We first modify the algorithm of Baum and Clausen to compute a full set \mathcal{F} of pairwise nonequivalent irreducible monomial and \mathcal{T} -adapted representations of LG , where L is a field extension of K containing a primitive e th root of unity, and a permutation γ of \mathcal{F} such that F^α is equivalent to γF : $F^\alpha \sim \gamma F$. Here α is the Frobenius automorphism of L/K . We then compute a full set \mathcal{F}' of representatives of $\text{Gal}(L/K)$ -orbits of \mathcal{F} and for each $F \in \mathcal{F}$ the degree of the character field of F over K .

Step 2. For each $F \in \mathcal{F}'$ we compute a realization T_F of F over its character field and then a realization of the trace of $T_F^{-1}FT_F$ over K .

Similar to the algorithm of Baum and Clausen, the arithmetics we use in these two steps consists just of symbolic computation in L^\times , where L is a field extension of K containing an e th root of unity. More precisely, we represent nonzero elements of L as integers i with $0 \leq i < |L| - 1$, where i corresponds to the element ω^i and ω is a fixed generator of L^\times . This representation of L allows to solve efficiently equations of the form $N(x) = \alpha$ or $x(x^\sigma)^{-1} = \alpha$, where $\alpha \in L$, N is the norm of L relative to a subfield M , and σ is the Frobenius automorphism of L/M . We shall need solutions to these kinds of equations in the second step of our algorithm. Moreover, as we will need primitive elements for subfields of L , this representation of L allows us to compute in advance these generators and store them in a list Ω .

The final step of the algorithm computes the KG -representations from the already computed realizations. This step requires matrix multiplication over L , and symbolic computation in L^\times does not suffice for this purpose. Strategies to solve this problem are discussed in the last section.

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2 Irreducibles of LG and $\text{Gal}(L/K)$ -Orbits

The first step of our algorithm takes as input a supersolvable group G in pc-presentation and a finite extension L of K containing a primitive e th root of unity; it outputs a list \mathcal{F} of pairwise inequivalent irreducible representations of LG and a permutation γ of \mathcal{F} such that $F^\alpha \sim \gamma F$, α being the Frobenius automorphism of L/K .

For the rest of this section we set $\mathcal{T}_i := (G_i > G_{i-1} > \cdots > G_0 = \{1\})$ for $1 \leq i \leq n$. In particular, $\mathcal{T} = \mathcal{T}_n$.

The algorithm of Baum and Clausen in [1] computes the list \mathcal{F} ; we modify this algorithm to obtain additional information on the orbits of \mathcal{F} under the action of the Galois group of L/K ; this information is encoded as the permutation γ .

Our algorithm works bottom up along \mathcal{T} . At level i , $1 \leq i \leq n$, it takes the following input:

- (1) \mathcal{F} , a full set of nonequivalent irreducible e -monomial representations of LG_{i-1} such that $\bigoplus_{F \in \mathcal{F}} F$ is \mathcal{T}_{i-1} -adapted;
- (2) For every $i-1 < j \leq n$ a permutation π_j of \mathcal{F} such that $F^{g_j} \sim \pi_j F$ for all $F \in \mathcal{F}$ as well as e -monomial matrices $X_{j,F} \in \text{Int}(F^{g_j}, \pi_j F)$, $F \in \mathcal{F}$;
- (3) A permutation γ of \mathcal{F} such that $F^\alpha \sim \gamma F$, as well as e -monomial matrices $M_F \in \text{Int}(F^\alpha, \gamma F)$, $F \in \mathcal{F}$;

and computes the following output:

- (1) \mathcal{D} , a full set of nonequivalent irreducible e -monomial representations of LG_i such that $\bigoplus_{D \in \mathcal{D}} D$ is \mathcal{T}_i -adapted;
- (2) For every $i < j \leq n$ a permutation τ_j of \mathcal{D} such that $D^{g_j} \sim \tau_j D$ for all $D \in \mathcal{D}$ as well as e -monomial matrices $Y_{j,D} \in \text{Int}(D^{g_j}, \tau_j D)$, $D \in \mathcal{D}$.
- (3) A permutation δ of \mathcal{D} such that $D^\alpha \sim \delta D$, as well as e -monomial matrices $N_D \in \text{Int}(D^\alpha, \delta D)$, $D \in \mathcal{D}$;

Outputs (1) and (2) are computed in exactly the same way as in the algorithm of Baum and Clausen [1]. Therefore, we only discuss the computation of Output (3) and assume that we have already performed the two phases of the algorithm in [1]. Note that during the construction at level i in Phase 1 there is built a bipartite graph in which $F \in \mathcal{F}$ and $D \in \mathcal{D}$ are linked iff F is a constituent of $D \downarrow G_{i-1}$. We will need this information to compute δ and N_D . For this we proceed in a similar way as does Phase 2 of the Baum-Clausen algorithm. Let $F \in \mathcal{F}$ and $p := [G_i : G_{i-1}]$. We distinguish two cases.

Case 1. $\pi_i F = F$, i.e., $F^{g_i} \sim F$. Since $(F^{g_i})^\alpha = (F^\alpha)^{g_i}$, we obtain

$$(\gamma F)^{g_i} \sim (F^\alpha)^{g_i} = (F^{g_i})^\alpha \sim F^\alpha \sim \gamma F.$$

We already know p extensions D_0, \dots, D_{p-1} of F and p extensions $\Delta_0, \dots, \Delta_{p-1}$ of γF . For $0 \leq k < p$ we have

$$D_k^\alpha \downarrow G_{i-1} = (D_k \downarrow G_{i-1})^\alpha = F^\alpha \sim \gamma F,$$

hence D_k^α is equivalent to one of the representations $\Delta_0, \dots, \Delta_{p-1}$. Thus there exists a permutation ρ of $\{0, \dots, p-1\}$ such that $D_k^\alpha \sim \Delta_{\rho k}$ for $0 \leq k < p$. Since $\text{Int}(D_k^\alpha, \Delta_{\rho k}) = \text{Int}(F^\alpha, \gamma F)$, we may set $N_{D_k} := M_F$. To determine δD_k , note that

$$M_F D_0^\alpha (g_i) M_F^{-1} = \Delta_\ell (g_i) = \chi^\ell (g_i G_{i-1}) \Delta_0 (g_i)$$

for a unique integer ℓ with $0 \leq \ell < p$. To compute ℓ , we just need to compare a nonzero entry of both sides of the above e -monomial matrix equation. We then set $\delta D_0 := \Delta_\ell$. For other values of k we can determine δD_k by cyclic shifts: $D_k^\alpha = (\chi^k \otimes D_0)^\alpha = (\chi^k)^\alpha \otimes D_0^\alpha \sim$

$(\chi^\alpha)^k \otimes (\chi^\ell \otimes \Delta_0)$. Hence $\delta D_k = \Delta_{(kq+\ell) \bmod p}$, since α is the Frobenius automorphism over $K = \mathbb{F}_q$.

Case 2. $\pi_i F \neq F$, i.e., $F^{g_i} \not\sim F$. In Phase 1 we have already constructed $D \in \mathcal{D}$ such that $D \downarrow G_{i-1} = \bigoplus_{k=0}^{p-1} F_k$ and $F_k = \pi_i^k F$ is of degree, say, f . Since $(F \uparrow G)^\alpha = F^\alpha \uparrow G$ and $F^\alpha \sim \gamma F$, δD is the unique representation $\Delta \in \mathcal{D}$ such that γF is an irreducible constituent of $\Delta \downarrow G_{i-1}$. According to our construction, $\Delta \downarrow G_{i-1} = \bigoplus_{k=0}^{p-1} \Phi_k$ with $\Phi_k = \pi_i^k \Phi$ for some $\Phi \in \mathcal{F}$. There is a permutation ρ of $\{0, \dots, p-1\}$ such that $\gamma F_k = \Phi_{\rho k}$ as well as e -monomial matrices $M_k := M_{F_k} \in \text{Int}(F_k^\alpha, \Phi_{\rho k})$. To compute $N_D \in \text{Int}(D^\alpha, \delta D)$, we consider $\text{Int}(D^\alpha \downarrow G_{i-1}, \delta D \downarrow G_{i-1})$. By Schur's Lemma there exist constants $d_0, \dots, d_{p-1} \in L^\times$ such that

$$N_D = (P_\rho \otimes I_f) \cdot \left(\bigoplus_{k=0}^{p-1} d_k M_k \right).$$

We may assume that $d_0 = 1$. To determine the other d_k , we use the equation

$$N_D D^\alpha(g_i) N_D^{-1} = (\delta D)(g_i). \quad (1)$$

According to our construction in Phase 1 there are e -monomial matrices $T_k, S_k \in L^{f \times f}$ such that

$$D(g_i) = (P_\pi \otimes I_f) \cdot \left(\bigoplus_{k=0}^{p-1} T_k \right)$$

and

$$(\delta D)(g_i) = (P_\pi \otimes I_f) \cdot \left(\bigoplus_{k=0}^{p-1} S_k \right),$$

where $\pi = (0, \dots, p-1)$. Hence, (1) is equivalent to

$$(P_\pi \otimes I_f) \cdot \left(\bigoplus_{k=0}^{p-1} d_{\pi k} M_{\pi k} \right) \cdot \left(\bigoplus_{k=0}^{p-1} T_k^\alpha \right) \cdot \left(\bigoplus_{k=0}^{p-1} d_k^{-1} M_k^{-1} \right) = (P_{\rho^{-1}\pi\rho} \otimes I_f) \cdot \left(\bigoplus_{k=0}^{p-1} S_{\rho k} \right).$$

Since $d_0 = 1$, we can successively determine d_1, \dots, d_{p-1} by comparing for each k one nonzero entry of $M_{\pi k} T_k^\alpha d_k^{-1} M_k^{-1}$ and $S_{\rho k}$.

We now compute a set \mathcal{F}' of representatives of $\text{Gal}(L/K)$ -orbits of \mathcal{F} and for each $F \in \mathcal{F}'$ with character χ_F the degree of the character field $d_F := [K(\chi_F):K]$ of F as well as a matrix $S_F \in \text{Int}(F^{\alpha^{d_F}}, F)$. (α^{d_F} generates the Galois group of $L/K(\chi_F)$.) Notice that $\ell := d_F$ is the smallest integer m such that $F^{\alpha^m} \sim F$, i.e., the smallest m such that $\gamma^m F = F$. Furthermore, it is easily checked that

$$S_F := M_{\gamma^{\ell-1}F} M_{\gamma^{\ell-2}F}^\alpha \cdots M_F^{\alpha^{\ell-1}} \in \text{Int}(F^{\alpha^\ell}, F).$$

The algorithm to compute the required data is now straight-forward. We take the first representation F in \mathcal{F} , append it to the list \mathcal{F}' , and set $M := M_F$. Then we go through all $\gamma^i F$, delete them from the list \mathcal{F} , update $M := M_{\gamma^i F} M^\alpha$, and stop as soon as $\gamma^\ell F$ equals F , deleting F from \mathcal{F} in this last step. In this way we also obtain d_F . We repeat the whole process until the list \mathcal{F} is empty.

3 Realization over Subfields

In this step of our algorithm we take the output of the last step and compute at first for each $F \in \mathcal{F}'$ a realization T_F of F over $K(\chi_F)$, where χ_F is the character of F . We then proceed by computing a realization of the trace of $T_F F T_F^{-1}$ over K .

It is well known that any absolutely irreducible representation of LG has a realization over its character field [3, Kapitel V, Satz 14.10]. We would like to give here a proof of this fact which builds the basis of our algorithm to find such a realization. We use the following setup: F is an irreducible representation of LG of degree f , M is the character field of F , $[L:M] =: \ell$, and β is a generator of $\text{Gal}(L/M)$. For a matrix $A \in L^{m \times m}$, we define the norm of A by $N_{L/M}(A) := A^{\beta^{\ell-1}} \cdots A$. Note that if $m \neq 1$, then the norm of A does not necessarily belong to $M^{m \times m}$.

The representations F and F^β are equivalent since they have the same character. Hence there exists an invertible matrix $S \in \text{Int}(F^\beta, F)$. Suppose that there exists $T \in \text{GL}(f, L)$ such that $T^{-1}T^\beta = S$. Then, $SFS^{-1} = F^\beta$ implies that TFT^{-1} is invariant under β , hence T is a realization of F over M . By Speiser's Theorem [7] mentioned in the introduction such a matrix T exists iff $N_{L/M}(S) = I_f$. (Speiser's original proof works only over infinite fields; for a general proof, see [6, page 151].) A straightforward calculation shows that $N_{L/M}(S) \in \text{Int}(F, F)$. Hence, Schur's Lemma implies that $N_{L/M}(S) = cI_f$ for some $c \in L$. But $N_{L/M}(S)^\beta = SN_{L/M}(S)S^{-1} = cI_f$, hence $c \in M$. Since L is finite, any element in M is the norm of an element in L , hence there exists $d \in L$ such that $N_{L/M}(dS) = I_f$. Replacing S by dS if necessary, we obtain the existence of T , a realization of F over its character field. (See also [3, Kapitel V, Bemerkung 14.14].)

From the second step we know $\ell := [L:K]/d_F$ and an e -monomial matrix $S = S_F \in \text{Int}(F^\beta, F)$, $\beta = \alpha^{d_F}$. Let $S =: P_\pi \text{diag}(S(1), \dots, S(f))$. We first compute some auxiliary data. Suppose that π can be written as the product of ν disjoint cycles of lengths ℓ_1, \dots, ℓ_ν and let ρ_1, \dots, ρ_ν be a complete set of disjoint representatives of each cycle. We compute ν , ℓ_1, \dots, ℓ_ν and ρ_1, \dots, ρ_ν , then a nonzero entry $\gamma := \prod_{j=0}^{\ell_i-1} S(\pi^j 1)^{\beta^{\ell_i-1-j}}$ of $N_{L/M}(S)$, and some element c of L satisfying $N_{L/M}(c) = \gamma^{-1}$; we then replace S by cS . Now we have $N_{L/M}(S) = I_f$. As ℓ_i divides the order of π and the latter divides ℓ , we have $\ell_i | \ell$. Hence, we can extract from the precomputed list Ω of primitive elements of subfields of L elements $y_1, \dots, y_\nu \in L$ such that y_i has degree ℓ_i over K . The rest of the algorithm, written in pseudo code, is now as follows (0^j means $0, \dots, 0$ j -times):

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2  for  $i = 1$  to  $\nu$  do
3       $\gamma_i := \prod_{j=0}^{\ell_i-1} S(\pi^j \rho_i)^{\beta^{\ell_i-1-j}}$ ;
4      Compute  $x_i \in L$  such that  $\gamma_i = x_i^{-1} x_i^{\beta^{\ell_i}}$ ;
5       $T[i] := (0^t, x_i, x_i y_i, \dots, x_i y_i^{\ell_i-1}, 0^{m-t-\ell_i})^\top$ ;
6      for  $j = 1$  to  $\ell_i - 1$  do
7           $T[\pi^j \rho_i] := S(\pi^{j-1} \rho_i)^{-1} T[\pi^{j-1} \rho_i]^\beta$ ;
8      od;
9       $t := t + \ell_i$ ;
10 od;
11  $T_F := (T[1] \mid T[2] \mid \cdots \mid T[f])$ .

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It is not clear in advance that the above algorithm is executable since there might be no element x_i satisfying the equation in line 4. In the following we show that such an x_i always exists and prove that the matrix T obtained by our algorithm is in fact a realization of F over M .

Let $T \in L^{f \times f}$ have columns $T[1], \dots, T[f]$. Then $TS = T^\beta$ iff for all $1 \leq i \leq \nu$ and all $1 \leq j \leq \ell_i$ we have

$$T[\pi^j \rho_i] = S(\pi^{j-1} \rho_i)^{-1} T[\pi^{j-1} \rho_i]^\beta. \quad (2)$$

This implies that $T[\rho_i] = \gamma_i^{-1} T[\pi^{\ell_i} \rho_i]^{\beta^{\ell_i}}$, hence $T[\rho_i] = N_{L/M_i}(\gamma_i)^{-1} T[\rho_i]^{\beta^{\ell_i}}$ which gives $N_{L/M_i}(\gamma_i) = 1$, where M_i is the fixed field of β^{ℓ_i} . Hence, by Hilbert's Theorem 90 there exists x_i satisfying the condition in line 4 and our algorithm is executable. Line 7 guarantees that (2) is satisfied for all $1 \leq i \leq \nu$, $1 \leq j < \ell_i$. To see that it is also satisfied for $j = \ell_i$, we only need to check that $T[\rho_i] = \gamma_i^{-1} T[\rho_i]^{\beta^{\ell_i}}$. But this follows from the choice of x_i and the fact that y_i is fixed under β^{ℓ_i} . It remains to show that T is invertible. This is true because the Vandermonde matrix $\left((y_i^{\beta^j})^k \right)_{0 \leq j, k \leq \ell_i - 1}$ is invertible (since y_i has degree ℓ_i over K).

At this stage of our algorithm we have a list \mathcal{F}' of representatives of $\text{Gal}(L/K)$ -orbits of the irreducible representations of LG , for each $F \in \mathcal{F}'$ the degree d_F of the character field of F over K , and a realization T_F of F over its character field. We know that $D_F := \bigoplus_{i=0}^{d_F-1} F^{\alpha^i}$ is equivalent to an irreducible representation of KG and that all irreducible representations of KG are obtained this way.

Let $F \in \mathcal{F}'$ be of degree f and $\tilde{F} := T_F F T_F^{-1}$. We extract from Ω a primitive element γ of the character field of F over K , i.e., an element having degree $d = d_F$ over K . Let $U := V \otimes I_f$, where V is the Vandermonde matrix $V := \left((\gamma^i)^{\alpha^j} \right)_{0 \leq i, j < d}$. It is easily verified that

$$R := U \cdot \begin{pmatrix} T_F & & & \\ & T_F^\alpha & & \\ & & \ddots & \\ & & & T_F^{\alpha^{d-1}} \end{pmatrix}.$$

is a realization of $D = \bigoplus_{i=0}^{d-1} F^{\alpha^i}$ over K .

4 The Final Step and Concluding Remarks

Given a finite field K , a supersolvable group G of exponent e in pc-presentation, and a field extension L of K containing a primitive e th root of unity, the first two steps of our algorithm have computed a set \mathcal{F}' of representatives of $\text{Gal}(L/K)$ -orbits of the irreducibles of LG , and for each such representation a realization of its trace over K . One possible strategy to compute the KG representations out of these data would be to represent L as the residue class ring modulo an irreducible polynomial, compute a primitive element ω of L^\times , replace each entry of the matrices involved by their corresponding polynomial representations, and proceed with matrix multiplication (and inversion) over L . Another strategy is to start with a representation of L as a polynomial residue class ring, and to go through all the steps of the algorithm using field arithmetic in L . Here we face the difficulty of solving equations of the type $x^d = \alpha$, where d is a divisor of $|L| - 1$. Both these strategies consume exponential

time, and it seems that in practice a correct implementation of any of these strategies is rather complicated.

Nevertheless, we have implemented our algorithm in the computer algebra system GAP [5]. In this implementation the final step is performed by using a table of Jacobi logarithms for L , which needs exponential space (and time). Although it is impractical for large $|L|$, this strategy performs well for small sizes of L .

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