On Valve Adjustments that Interrupt all s-t-Paths in a Digraph

U l r i c h H u c k e n b e c k [1)]

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Abstract: When searching a path in a digraph, usually the following situation is given: Every node v may be entered by an arbitrary incoming arc (u, v), and v may be left by an arbitrary outgoing arc (v, w).

In this paper, however, we consider graphs with valve nodes, which cannot arbitrarily be entered and left. More precisely, a movable valve is installed in each valve node v. Entering v via (u, v) and leaving it via (v, w) is only possible if the current position of the valve generates a connection between these two arcs; if, however, the current valve adjustment interrupts this connection then every path using the arcs (u, v) and (v, w) is interrupted, too.

We investigate the complexity of the following problem

> Given a digraph with valve nodes. Let s and t be two nodes of this graph.
> Does there exist a valve adjustment that interrupts all paths from s to t?

We show that this problem can be solved in deterministic polynomial time if all valve nodes belong to a particular class of valves; otherwise the problem is $\mathcal{NP}$-complete.

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1. Introduction

One of the most prominent problems in graph theory is the search of paths in digraphs. Usually, the following assumption is made: A path may use any arbitrary incoming and outgoing arc when entering and leaving a node, resp.

In this paper, however, the arcs used by a path must satisfy particular conditions. More precisely, it is assumed that valves are intalled in several nodes of the given network (see *Figure 1*). A path may only use arcs that are connected by the current valve adjustment.

This setting is of great practical relevance. E.g., Braun's diploma thesis [1] is about systems of pipes in breweries where several crossings of pipes are equipped with valves. Another realization of valve graphs are processing networks; connecting particular incoming and outgoing wires of a processing element (PE) can be interpreted as adjusting a valve; in particular, a faulty PE can be simulated by connecting an incoming arc with an outgoing arrow that ends in a dead end (see *Figure 1c)*).

Both papers [1] and [2] are about the existence of a valve adjustment that allows a path P from a node s to another node t. Braun gives a heuristic algorithm to find such a path P with optimal length; in [2], however, it is shown an exact solution of this problem is $\mathcal{NP}$-complete for a great class of valves.

Here, we consider the inverse question: Is it possible to find a blocking valve adjustment ϑ, i.e. ϑ allows *no* path from s to t?
This problem has several interesting practical and theoretical aspects:

- Consider s and t as processing elements; then *all* "valve" adjustments should allow a connection from s to t; hence, the existence of a blocking valve adjustment means that the quality of the network is not high enough.
  If simulating faulty PE's by valves then the existence of a blocking valve adjustment means a bad fault tolerance of the network.
- We can define a new type of connected components: Two nodes a, b are called *strictly connected* ($a \sim b$) if there exists no blocking valve adjustment for a, b or for b, a; i.e. all valve adjustments allow an a-b-path and a b-a-path. Then $\sim$ is an equivalence relation whose equivalence classes can be considered as connected components.

We show that the problem of blocking valve adjustments can be solved in deterministic polynomial time if all valve nodes are in a restricted class $\mathbf{T}^\bullet$; otherwise the problem is $\mathcal{NP}$-complete.

2. Basic Notations and Definitions

Definition 2.1.
The set of all *natural numbers* is $\mathbb{N} := \{1, 2, 3, \ldots\}$; we write $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. — If X, Y are two arbitrary sets then $X \times Y$ is their *cartesian product*. Moreover, $|X|$ is the *cardinality* of X. At last, the set 2^X consists of all subsets of X. ∎

Definition 2.2.
A *digraph (without parallels and loops)* is a pair $\mathcal{G} = (\mathcal{V}, \mathcal{R})$ of sets where $\mathcal{V}$ is the set of *nodes* and $\mathcal{R} \subseteq (\mathcal{V} \times \mathcal{V}) \backslash \{(v, v) \mid v \in \mathcal{V}\}$ is the set of *arcs*.
For any arc $r = (u, v) \in \mathcal{R}$ we write $\alpha(r) := u$ and $\omega(r) := v$. Moreover, u and v are called *incident with* r.

If $\mathcal{G} = (\mathcal{V}, \mathcal{R})$ is directed and $v \in \mathcal{V}$ then we define the sets $\mathcal{R}^-(v)\,, \mathcal{R}^+(v) \subseteq \mathcal{R}$ as $\mathcal{R}^-(v) := \{(x, y) \in \mathcal{R} \mid y = v\}$ and $\mathcal{R}^+(v) := \{(x, y) \in \mathcal{R} \mid x = v\}$. The quantities $g^-(v) := |\mathcal{R}^-(v)|$ and $g^+(v) := |\mathcal{R}^+(v)|$ are the *indegree* and the *outdegree* of v, resp.

Moreover, $S^-(\mathcal{V}) := \{v \in \mathcal{V} \mid g^-(v) = 0\}$ is the set of all *sources*, and $S^+(\mathcal{V}) := \{v \in \mathcal{V} \mid g^+(v) = 0\}$ is the set of all *sinks*. ∎

Remark 2.3. Throughout our paper we only consider *finite* graphs. ∎

Definition 2.4. *(Paths in Graphs)*

Given a digraph $\mathcal{G}$.

A *path* in $\mathcal{G}$ is a sequence $P = [x_0, \ldots, x_l]$ with $(x_\lambda, x_{\lambda+1}) \in \mathcal{R}$ for all $\lambda = 0, \ldots, l-1$. We write $\alpha(P) := x_0$ and $\omega(P) := x_l$.

If $x_0 = x_l$ then P is called a *cycle*. We say that P is *elementary* iff all nodes $x_0, \ldots, x_{l-1}$ are pairwise distinct and $x_l \notin \{x_1, \ldots, x_{l-1}\}$; in particular, elementary cycles with $x_0 = x_l$ are possible. (The definition of elementary paths is analogous to [3], p. 28.)

If $P = [v_0, \ldots, v_l]$ is given then any path $Q = [v_0, \ldots, v_\lambda]$ with $\lambda \leq l$ is called a *prefix* of P ("$Q \leq P$"). If $Q = [v_{\lambda'}, v_{\lambda'+1}, \ldots, v_{\lambda''}]$ with $0 \leq \lambda' \leq \lambda'' \leq \lambda$ then Q is a *subpath* of P; this is written as $Q \subseteq P$. In particular, P is a prefix and a subpath of itself.

Given two paths Q and Q' with $\alpha(Q') = \omega(Q)$. Then the *concatenation* $P := Q \oplus Q'$ is the path that first uses Q and then traverses Q'. The operation '$\oplus$' is also defined for arcs; e.g., the path $P = Q \oplus r \oplus r' \oplus Q'$ uses the path Q, the arcs r, r' and the path Q' in this order.

The set of all paths in $\mathcal{G}$ is written as $\mathcal{P}(\mathcal{G})$. Moreover, if $v, w \in \mathcal{V}$ then $\mathcal{P}(v)$ is the set of all paths starting from v, and $\mathcal{P}(v, w)$ contains all paths from v to w; every element of $\mathcal{P}(v, w)$ is called a *v-w-path*. ∎

3. The Definition of Valve Graphs

We consider the following situation: A valve is installed in some nodes v of a digraph $\mathcal{G}$; each valve adjustment connects exactly one incoming arc $r^- \in \mathcal{R}^-(v)$ with some outgoing arcs $r^+ \in \mathcal{R}^+(v)$. This corresponds to the requirement that a stream of liquid may enter a valve only by one incoming pipe.

Definition 3.1. A *(directed) valve graph* is a triple $\mathcal{H} = (\mathcal{G}, \widehat{\mathcal{V}}, \gamma)$ with the following properties:

$\mathcal{G} = (\mathcal{V}, \mathcal{R}, \alpha, \omega)$ is a digraph. The set $\widehat{\mathcal{V}} \subseteq \mathcal{V} \backslash (S^-(\mathcal{G}) \cup S^+(\mathcal{G}))$ contains all *valve nodes* of $\mathcal{G}$. The third component is the *valve function* γ; for every arc (u, v) with $v \in \widehat{\mathcal{V}}$, the set $\gamma(u, v)$ consists of all arcs (v, w) which can be used after (u, v). E.g., $\gamma(r_1^-) = \{r_1^+, r_2^+\}$ in *Fig. 1*. More formally, $\gamma : \bigcup_{v \in \widehat{\mathcal{V}}} \mathcal{R}^-(v) \longrightarrow 2^{\mathcal{R}}$ with the following properties:

($) $\gamma(u, v) \subseteq \mathcal{R}^+(v)$ for all $v \in \widehat{\mathcal{V}}$ and $(u, v) \in \mathcal{R}$.

($$) For every node $v \in \widehat{\mathcal{V}}$ and every arc $(v, w) \in \mathcal{R}$ there exists an arc $(u, v) \in \mathcal{R}$ such that $(v, w) \in \gamma(u, v)$.

The last condition means that every outgoing arc (v,w) can be reached from at least one incoming arc (u,v).

Given a valve graph $\mathcal{H} = (\mathcal{G}, \widehat{\mathcal{V}}, \gamma)$. A *valve adjustment* is a function $\vartheta : \widehat{\mathcal{V}} \to \mathcal{R}$ such that $\vartheta(v) \in \mathcal{R}^-(v)$ for all $v \in \widehat{\mathcal{V}}$. This means that the arc $\vartheta(v)$ is connected with all arcs $r^+ \in \gamma(\vartheta(v))$; moreover, each connection between any further incoming arc $r^- \neq \vartheta(v)$ and any outgoing arc $r \in \mathcal{R}^+(v) \backslash \gamma(\vartheta(v))$ is interrupted.

Given a valve adjustment ϑ and a path $P \in \mathcal{P}(\mathcal{G})$. Then P is called *ϑ-admissible* or *ϑ-legal* iff P can be realized by ϑ; this means that every valve node v visited by P is entered via $\vartheta(v)$ and left by an arc $r^+ \in \gamma(\vartheta(v))$.
A path P is called *admissible* or *legal* iff there exists a valve adjustment ϑ such that P is ϑ-admissible. ■

We next describe the structure of valve nodes. Perhaps the following definition is somewhat formal and abstract; but it allows correct description of *sets* of valve types. The underlying idea is that particular valves in a graph are isomorphic copies of a given prototype τ of a valve.

Definition 3.2.

a) A *valve type* or *normalized valve node* is a triple $< k^-, k^+, f >$ where $k^-, k^+ \in \mathbb{N}$. The function f says which of the outgoing arcs $1, \ldots, k^+$ are connected with a given incoming arc $\kappa \in \{1, \ldots, k^-\}$. More precisely, f is defined as $f : \{1, \ldots, k^-\} \to 2^{\{1,\ldots,k^+\}} \backslash \emptyset$ such that

$$(++) \qquad \bigcup_{\kappa^- = 1}^{k^-} f(\kappa^-) = \{1, \ldots, k^+\}\,.$$

This condition is analogous to (\$\$) in *Definition 3.1* and means that each outgoing arc κ^+ can be reached by an incoming arc κ^-.
An example can be seen in *Figure 2* where $\tau = \langle 2, 2, f\rangle$ with $f(1) = \{1\}$ and $f(2) = \{2\}$.
The set of all valve types $\tau = \langle k^-, k^+, f\rangle$ is defined as $\mathbf{T}$.

b) A normalized valve $\tau = \langle k^-, k^+, f\rangle$ is *complete* if $f(\kappa^-) = \{1, \ldots, k^+\}$ for all $\kappa^- = 1, \ldots, k^-$. This means that each outgoing arc can be reached from any given incoming one. The set of these valve types is $\mathbf{T}_0$. All further valve types are called *incomplete* and form the set $\mathbf{T}_1 := \mathbf{T} \backslash \mathbf{T}_0$.

c) Given a valve graph $\mathcal{H} = (\mathcal{G}, \widehat{\mathcal{V}}, \gamma)$ and $v \in \widehat{\mathcal{V}}$. Then we say that v is of type $\tau = \langle g^-(v), g^+(v), f\rangle$ is v a "copy" of τ (up to isomorphy); more precisely, we require that there exist bijections $\eta : \mathcal{R}^-(v) \to \{1, \ldots, g^-(v)\}$ and $\chi : \mathcal{R}^+(v) \to \{1, \ldots, g^+(v)\}$ with the following property: The fact that some arc $r^- \in \mathcal{R}^-(v)$ is connected with $r^+ \in \mathcal{R}^+(v)$ is equivalent to a connection from $\eta(r^-)$ to $\chi(r^+)$ in the normalized valve nodes. More formally,

$$(*) \quad \big(\forall\ r^- \in \mathcal{R}^-(v)\,,\ r^+ \in \mathcal{R}^+(v)\big) \quad r^+ \in \gamma(r^-) \iff \chi(r^+) \in f(\eta(r^-))\,.$$

$$\left(\begin{array}{l}\text{Another formulation of } (*) \text{ is :}\\ \gamma(r^-) \;=\; \big\{\chi^{-1}(\kappa^+)\,\big|\,\kappa^+ \in f(\eta(r^-))\big\} \text{ for all } r^- \in \mathcal{R}^-(v)\,.\end{array}\right)$$

For example, recall *Figure 1* and let $\eta(r_i^-) := i$ and $\chi(r_j^+) := j$ $(i,j = 1,2,3,4)$. Then v is of type $\langle 3, 2, f\rangle$ where $f(1) = \{1,2\}$, $f(2) = \{2\}$ and $f(3) = \{3\}$.

The fact that v is of type τ is abbreviated as $v \rightsquigarrow \tau$. Note that v can be of different types $\langle g^-(v), g^+(v), f_1\rangle \neq \langle g^-(v), g^+(v), f_2\rangle$ because the enumerations η and χ can be changed.

d) Let $T \subseteq \mathbf{T}$. Then $(\mathcal{G}, \widehat{\mathcal{V}}, \gamma)$ is called a *T-graph* iff for every $v \in \mathcal{V}$ there exists a $\tau \in T$ such that v is of type T. In this case we write $(\mathcal{G}, \widehat{\mathcal{V}}, \gamma) \rightsquigarrow T$ or $\mathcal{G} \rightsquigarrow T$. ■

4. The Problem of Blocking Valve Adjustments

We now formulate the <u>P</u>roblem of <u>B</u>locking <u>V</u>alve <u>A</u>djustments for <u>d</u>igraphs $\big(\mathbf{PBVA}_d(T)\big)$ where $T \subseteq \mathbf{T}$ is fixed set of valve types:

Given a valve graph $\mathcal{H} = \left(\mathcal{G}, \widehat{\mathcal{V}}, \gamma\right) \rightsquigarrow T$ *and two nodes* $s \in S^-(\mathcal{V})$, $t \in S^+(\mathcal{V})$. *Does there exist a valve adjustment such that there is no* ϑ*-admissible* s-t*-path* P*?*

Our investigation starts with a special case: Let $\tau = \langle 2, 2, f^* \rangle$ with $f^*(i) := \{i\}$, $i = 1, 2$ (see *Figure 2.*) Then the following result is true:

Theorem 4.1. The problem $\mathbf{PBVA}_d(\{\tau^*\})$ is $\mathcal{NP}$-complete.

Proof: It is clear that this problem can be solved in nondeterministic polynomial time: If $\mathcal{G}$ is given then the Turing machine guesses a valve adjustment and tests whether t cannot be reached from s.

We next show that $\mathbf{PBVA}_d(\{\tau^*\})$ is $\mathcal{NP}$-hard. For this we reduce 3-SAT to it. Given a boolean formula $C = C(1) \cdot \ldots \cdot C(k)$ where each clause $C(\kappa)$ is of the form $C(\kappa) = (u_1(\kappa) + u_2(\kappa) + u_3(\kappa))$ with $u_1(\kappa), u_2(\kappa), u_3(\kappa) \in \{x_1, \ldots, x_n, \overline{x}_1, \ldots, \overline{x}_n\}$; we must decide whether there exists a function $f : \{x_1, \ldots, x_n\}$ verifying C.

The first step of the reduction is considering the negation $T := \neg C$. T has the structure $T = T(1) + \ldots + T(k)$, where $T(\kappa) = z_1(\kappa) \cdot z_2(\kappa) \cdot z_3(\kappa)$ with $z_1(\kappa), z_2(\kappa), z_3(\kappa) \in \{x_1, \ldots, x_n, \overline{x}_1, \ldots, \overline{x}_n\}$ for all $\kappa = 1, \ldots, k$. Here we must test whether there exists a function $f : \{x_1, \ldots, x_n\}$ falsifying T or whether T is a tautology.

For this purpose we define the valve graph $\mathcal{H} = (\mathcal{G}, \widehat{V}, \gamma)$. Its construction is divided into two steps. We first generate the subgraphs $\mathcal{G}_\nu$ representing the literals x_ν and $\overline{x}_\nu$, $\nu = 1, \ldots, n$. Then we complete the graph $\mathcal{G}$.

The global structure of this construction is quite similar to that of *Theorem 3.4* in [2] In both proofs, the fact that all valves of $\mathcal{G}_\nu$ are in horizontal and vertical position corresponds to $f(x_\nu) = 1$ and $f(x_\nu) = 0$, resp.; these two valve adjustments of $\mathcal{G}_\nu$ are called *consistent*.

It is our aim to restrict our attention to consistent valve adjustments. So the graph $\mathcal{G}$ in *Theorem 3.4* of [2] is constructed such that

(A) C is satisfiable iff
(B) There exists some valve adjustment allowing an s-t-path iff
(C) There exists a *consistent* valve adjustment allowing an s-t-path.

But the assertion $(B) \Rightarrow (C)$ is not self evident; it is effected by the construction of the graphs $\mathcal{G}_\nu$; roughly spoken, the inconsistent adjustments allow *at most* the paths given by consistent ones, and this is effected by a "series connection" of valves.

In the current situation we desire the following situation:

(A') $T = \neg C$ is a tautology iff
(B') There exists *no* blocking valve adjustment, i.e. every valve adjustment allows an s-t-path iff
(C') Every *consistent* valve adjustment allows an s-t-path.

Here $(C') \Rightarrow (B')$ is non-trivial and must be made true by the construction of the graphs $\mathcal{G}_\nu$. This means that the non-consistent valve adjustments must admit *at least* the paths that are possible by consistent ones.

This is the reason why copying the graphs $\mathcal{G}_\nu$ on page 94 of [2] is not good in this situation. We need another construction basing of "parallel connection" of valves. The construction of $\mathcal{G}$ takes place in a cartesian coordinate system; the notation $A =_x B$ and $A =_y B$ means that the points A, B have the same x- and y-coordinate, resp. An example of $\mathcal{G}$ is *Figure 3*.

Let us first describe the graphs $\mathcal{G}_\nu$; they have thick arrows in *Figure 3*. We define the sets of all κ for which $T(\kappa)$ contains the factor x_ν and $\overline{x}_\nu$, resp.:

$$X_\nu := \{\kappa \mid x_\nu \text{ occurs in } T(\kappa)\}; \qquad \overline{X}_\nu := \{\overline{\kappa} \mid \overline{x}_\nu \text{ occurs in } T(\overline{\kappa})\}.$$

We assume that neither X_ν nor $\overline{X}_\nu$ is empty; otherwise we add a dummy term $x_\nu \cdot \overline{x}_\nu \cdot \overline{x}_{\nu'}$ with $\nu \neq \nu'$ to T. For every $\kappa \in X_\nu$ we generate a start node $s_\nu(\kappa)$ and an end note $t_\nu(\kappa)$. We draw these nodes such that

$s_\nu(\kappa) =_y t_\nu(\kappa)$, $s_\nu(\kappa) =_x s_\nu(\kappa')$ and $t_\nu(\kappa) =_x t_\nu(\kappa')$ for all $\kappa, \kappa' \in X_\nu$, where $t_\nu(\kappa)$ lies on the *right* of $s_\nu(\kappa)$. For every $\overline{\kappa} \in \overline{X}_\nu$ we generate a start node $\overline{s}_\nu(\overline{\kappa})$ and an end note $\overline{t}_\nu(\overline{\kappa})$. We draw these nodes such that $\overline{s}_\nu(\overline{\kappa}) =_x \overline{t}_\nu(\overline{\kappa})$, $\overline{s}_\nu(\overline{\kappa}) =_y \overline{s}_\nu(\overline{\kappa}')$ and $\overline{t}_\nu(\overline{\kappa}) =_y \overline{t}_\nu(\overline{\kappa}')$ for all $\overline{\kappa}, \overline{\kappa}' \in \overline{X}_\nu$, where $\overline{t}_\nu(\overline{\kappa})$ lies *under* $\overline{s}_\nu(\overline{\kappa})$.
After this we draw a valve node $v_\nu(\kappa, \overline{\kappa})$ for each pair $(\kappa, \overline{\kappa}) \in X_\nu \times \overline{X}_\nu$; moreover, we generate the following arcs:

$$\Big(s_\nu(\kappa), v_\nu(\kappa,\overline{\kappa})\Big) \ , \quad \Big(v_\nu(\kappa,\overline{\kappa}), t_\nu(\kappa)\Big) \ , \qquad \Big(\overline{s}_\nu(\overline{\kappa}), v_\nu(\kappa,\overline{\kappa})\Big) \ , \quad \Big(v_\nu(\kappa,\overline{\kappa}), \overline{t}_\nu(\overline{\kappa})\Big) \ .$$

So we obtain the *horizontal* paths $P_\nu(\kappa,\overline{\kappa}) := \Big[s_\nu(\kappa), v_\nu(\kappa,\overline{\kappa}), t_\nu(\kappa)\Big]$ and the *vertical* paths $\overline{P}_\nu(\kappa,\overline{\kappa}) := \Big[\overline{s}_\nu(\overline{\kappa}), v_\nu(\kappa,\overline{\kappa}), t_\nu(\overline{\kappa})\Big]$ where $\kappa \in X_\nu$ and $\overline{\kappa} \in \overline{X}_\nu$.

Now the construction of $\mathcal{G}_\nu$ is finished, and we must complete $\mathcal{G}$ (see *Figure 3*). To make its description easier we introduce the following notation: The exponent "1" means "no bar", the exponent "-1" means a bar; e.g., $x_\nu^1 = x_\nu$, $s_\nu^{-1}(i) = \overline{s}_\nu(i)$, $t_\nu^1(\kappa) = t_\nu(\kappa)$ and $P_\nu^{-1}(\kappa,\overline{\kappa}) = \overline{P}_\nu(\kappa,\overline{\kappa})$.

We generate the start node s and the end node t. Then we draw the following arrows:

a) Arcs from s to the start nodes indicated by $z_1(i),\ i = 1, \dots, k$:
FOR ALL $i = 1, \dots, k$ DO
Let $z_1(i) = x_\nu^\alpha$ where $\alpha \in \{\pm 1\}$, $\nu \in \{1, \dots, n\}$; then generate the arc $\Big(s, s_\nu^\alpha(i)\Big)$.

b) Arcs from the end nodes corresponding to $z_l(i)$ to the start nodes representing $z_{l+1}(i)$ $(i = 1, \dots, k,\ \ l = 1, 2)$:
FOR ALL $l = 1, 2$ DO
FOR ALL $i = 1, \dots, k$ DO
Let $z_l(i) = x_\mu^\alpha$ and $z_{l+1}(i) = x_\nu^\beta$ where $\alpha, \beta \in \{\pm 1\}$, $\mu, \nu \in \{1, \dots, n\}$;
then generate the arc $\Big(t_\mu^\alpha(i), s_\nu^\beta(i)\Big)$;

c) Arcs from the end nodes indicated by $z_3(i)$ to t, $i = 1, \dots, k$:
FOR ALL $i = 1, \dots, k$ DO
Let $z_3(i) = x_\nu^\beta$ where $\beta \in \{\pm 1\}$, $\nu \in \{1, \dots, n\}$; then generate the arc $\Big(t_\nu^\beta(i), t\Big)$.

Now the construction of $\mathcal{G}$ is finished, and we must show the equivalence of the following assertions:

(A) T is a tautology. (B) All valve adjustments of $\mathcal{G}$ allow an s-t-path.

We first show $(A) \Rightarrow (B)$ and start with an important observation:

(1) For all $\nu = 1, \dots, n$ the following is true:
Given an arbitrary adjustment of the valves $v_\nu(\kappa, \overline{\kappa})$, $\kappa \in X_\nu, \overline{\kappa} \in \overline{X}_\nu$. Then

(1.1) Every $t_\nu(\kappa)$, $\kappa \in X_\kappa$ can be reached from $s_\nu(\kappa)$ or

(1.2) every $\overline{t}_\nu(\overline{\kappa})$, $\overline{\kappa} \in \overline{X}_\kappa$ can be reached from $\overline{s}_\nu(\overline{\kappa})$.

To prove this we assume that (1.1) is not true. Then there exists a κ' such that all paths $P_\nu(\kappa', \overline{\kappa})$ are blocked; this means that all valves $v_\nu(\kappa', \overline{\kappa})$, $\overline{\kappa} \in \overline{X}_\nu$, are in vertical position. Hence all paths $\overline{P}_\nu(\kappa', \overline{\kappa})$, $\overline{\kappa} \in \overline{X}_\nu$, are admissible connections between $\overline{s}_\nu(\overline{\kappa})$ and $\overline{t}_\nu(\overline{\kappa})$.

Given now a valve adjustment ϑ. We must show that it admits an s-t-path. For this we define $f(x_\nu) := 1$ if (1.1) is true for $\mathcal{G}_\nu$ and $f(x_\nu) := 0$ otherwise. Since T is a tautology we can find an $i \in \{1, \dots, k\}$ such that $z_1(i), z_2(i), z_3(i)$ are made true by f. Let $z_l(i) = x_{\nu_l}^{\alpha_l}$ where $\alpha_l \in \{\pm 1\}$ and $\nu_l \in \{1, \dots, n\}$ for all $l = 1, 2, 3$. Then for all l the following is true:
If $\alpha_l = 1$ then $f(x_{\nu_l}) = 1$ because f verifies $T(i)$; this means that (1.1) is true for $\mathcal{G}_{\nu_l}$. Hence there exists an admissible path $Q_l(i)$ from $s_{\nu_l}(i) = s_{\nu_l}^{\alpha_l}(i)$ to $t_{\nu_l}(i) = t_{\nu_l}^{\alpha_l}(i)$.
If, however $\alpha_l = -1$ then $f(x_{\nu_l}) = 0$; this means that (1.1) is false, but then (1.2) must be true. So there exists an admissible path $Q_l(i)$ from $\overline{s}_{\nu_l}(i) = s_{\nu_l}^{\alpha_l}(i)$ to $\overline{t}_{\nu_l}(i) = t_{\nu_l}^{\alpha_l}(i)$.
In any case the valve adjustment allows the s-t-path

$$P \ := \ \Big(s, s_{\nu_1}^{\alpha_1}(i)\Big) \oplus Q_1(i) \oplus \Big(t_{\nu_1}^{\alpha_1}(i), s_{\nu_2}^{\alpha_2}(i)\Big) \oplus Q_2(i) \oplus \Big(t_{\nu_2}^{\alpha_2}(i), s_{\nu_3}^{\alpha_3}(i)\Big) \oplus Q_3(i) \oplus \Big(t_{\nu_3}^{\alpha_3}(i)\,,\,t\Big)\ .$$

We next show $\neg(A) \Rightarrow \neg(B)$. If T is not a tautology then there exists a function $f : \{x_1, \dots, x_n\} \to \{0, 1\}$ falsifying T. Then choose the valve adjustment ϑ such that all valves of $\mathcal{G}_\nu$, $\nu = 1, \dots, n$, are in horizontal and vertical position if $f(x_\nu) = 1$ and $f(x_\nu) = 0$, resp. Consequently, for all $\nu = 1, \dots, n$ the following is true:

(2) If $f(x_\nu) = 1$ then there is no ϑ-admissible connection from any $\overline{s}_\nu(\overline{\kappa})$ to $\overline{t}_\nu(\overline{\kappa})$, $\overline{\kappa} \in \overline{X}_\nu$.
If $f(x_\nu) = 0$ then there is no ϑ-admissible connection from any $s_\nu(\kappa)$ to $t_\nu(\kappa)$, $\kappa \in X_\nu$.

Let us now try to find a ϑ-admissible s-t-path P. It is easy to see that all nodes $s_\nu(i)$, $\overline{s}_\nu(i)$, $t_\nu(i)$, $\overline{t}_\nu(i)$ occurring along P must have the same argument $i \in \{1, \ldots, k\}$. So the first step of generating P is choosing i.
When this is done we consider the term $T(i) = z_1(i) \cdot z_2(i) \cdot z_3(i)$ where $z_l(i) = x_{\nu_l}^{\alpha_l}$ for all $l = 1, 2, 3$. As f refutes T there exists an l' such that

$$\Big((a) \quad \alpha_{l'} = 1 \text{ and } f(x_{\nu_{l'}}) = 0 \Big) \qquad \text{or} \qquad \Big((b) \quad \alpha_{l'} = -1 \text{ and } f(x_{\nu_{l'}}) = 1 \Big) .$$

In case (a), the path P must visit $s_{\nu_{l'}}(i)$ and $t_{\nu_{l'}}(i)$, but these nodes are disconnected because of (2) and $f(s_{\nu_{l'}}) = 0$.
In case (b), the path P must visit $\overline{s}_{\nu_{l'}}(i)$ and $\overline{t}_{\nu_{l'}}(i)$, but these nodes are not connected because of (2) and $f(s_{\nu_{l'}}) = 1$.
So the construction of a ϑ-admissible s-t-path fails. ∎

After considering this important special case we next investigate general classes T of valve types. In analogy to *Definition 3.2 b)* of [2] we introduce a set $\mathbf{T}^\bullet$ with the following property: If $T \subseteq \mathbf{T}^\bullet$ then $\mathbf{PBVA}_d(T)$ can be solved in deterministic polynomial time; otherwise it is $\mathcal{NP}$-complete.

Definition 4.2.

a) Given an abstract valve $\tau = \langle k^-, k^+, f \rangle$. We say that $\kappa^- \in \{1, \ldots, k^-\}$ *has a minimal set of connections* if $f(\kappa^-)$ is minimal with respect to subsets; this means that there is no $\lambda^- \neq \kappa^-$ with $f(\lambda^-) \subsetneq f(\kappa^-)$.

b) $\tau = \langle k^-, k^+, f \rangle$ has a *unique minimal set of connections* if $f(\kappa^-) = f(\lambda^-)$ for all κ^-, λ^- with a minimal set of connections.
An example is given in *Fig. 4,* where a valve type $\tau^{**} = \langle 2, 2, f^{**} \rangle$ with $f^{**}(1) = \{1\}$ and $f^{**}(2) = \{1, 2\}$ is shown. It is interesting to compare the result *4.5* and *Theorem 3.5* in [2] both of them having to do with τ^{**}.

c) The set of all τ described in *b)* is defined as $\mathbf{T}^\bullet$. ∎

The following result describes the structure of valves with unique minimal set of connections:

Lemma 4.3. Let $\tau = \langle k^-, k^+, f \rangle \in \mathbf{T}^\bullet$ and let k_τ^- be an entrance of τ such that $f(k_\tau^-)$ is minimal. Then

a) $(\forall\, \kappa = 1, \ldots, k^-)\ f(k_\tau^-) \subseteq f(\kappa)$. b) $\bigcap_{\kappa=1}^{k^-} f(\kappa) = f(k_\tau^-)$. c) $\bigcap_{\kappa=1}^{k^-} f(\kappa) \neq \emptyset$.

We first prove <u>*Assertion a)*</u>. For every $\kappa \in \{1, \ldots, k^-\}$ there exists a minimal subset $f(\lambda) \subseteq f(\kappa)$ (possibly $\lambda = \kappa$). The uniqueness of this minimal set implies that $f(k_\tau^-) = f(\lambda)$.
The proof to <u>*Assertion b)*</u> is very simple: First $\bigcap_{\kappa=1}^{k^-} f(\kappa) \subseteq f(k_\tau^-)$ as $f(k_\tau^-)$ is one of the members of the intersection. On the other hand, the assertion in *Part a)* implies that $\bigcap_{\kappa=1}^{k^-} f(\kappa) \supseteq f(k_\tau^-)$.
At last, <u>*Assertion c)*</u> is an immediate consequence of *Part b)* and $f(k_\tau^-) \neq \emptyset$. ∎

We next show that the problem of blocking valve adjustment for T-graphs is $\mathcal{NP}$-complete if not $T \subseteq \mathbf{T}^\bullet$. If, however, $\mathbf{T}$ is a subset of $\mathbf{T}^\bullet$ then our question can be answered in deterministic polynomial time.

Theorem 4.4. Let $T \not\subseteq \mathbf{T}^\bullet$. Then $\mathbf{PBVA}_d(T)$ is NP-complete.
This is even the case if $T = \{\tau\}$ where $\tau = \langle k^-, k^+, f \rangle$ does not have an unique minimal set of connections.

<u>*Proof:*</u> It is clear that our problem can be solved in nondeterministic polynomial time. To see the $\mathcal{NP}$-hardness we show that the problem $\mathbf{PBVA}_d(\{\tau^*\})$ can be reduced to the given problem $\mathbf{PBVA}_d(\{\tau\})$. This is can be seen in *Figure 5 – 7*.

We first note that there are two entrances $\kappa_\tau^-, \lambda_\tau^-$ of τ with different minimal sets $f(\kappa_\tau^-) \neq f(\lambda_\tau^-)$. An example is given in *Figure 5*. The fact that $f(\kappa_\tau^-) \neq f(\lambda_\tau^-)$ and the minimality of $f(\kappa_\tau^-)$ and $f(\lambda_\tau^-)$ imply that neither $f(\kappa_\tau^-) \subseteq f(\lambda_\tau^-)$ nor $f(\kappa_\tau^-) \supseteq f(\lambda_\tau^-)$ so that

(1) $f(\kappa_\tau^-) \backslash f(\lambda_\tau^-) \neq \emptyset$ and $f(\lambda_\tau^-) \backslash f(\kappa_\tau^-) \neq \emptyset$.

So we can describe the structure of τ:

(2) For each $\kappa = 1, \ldots, k^-$ exactly one of the following assertions is true:

(2.1) $\kappa = \kappa_\tau^-$, and κ is connected with an exit $\phi(\kappa) \in f(\kappa)$ such that $\phi(\kappa) \in f(\kappa_\tau^-) \backslash f(\lambda_\tau^-)$.

(2.2) $\kappa = \lambda_\tau^-$, and κ is connected with an exit $\phi(\kappa) \in f(\kappa)$ such that $\phi(\kappa) \in f(\lambda_\tau^-) \backslash f(\kappa_\tau^-)$.

Fig. 5

(2.3) $\kappa \notin \{\kappa_\tau^-, \lambda_\tau^-\}$, and κ is connected with an exit $\phi(\kappa) \in f(\kappa)$ such that $\phi(\kappa) \in f(\kappa_\tau^-) \backslash f(\lambda_\tau^-)$.

(2.4) $\kappa \notin \{\kappa_\tau^-, \lambda_\tau^-\}$, (2.3) is false, and κ is connected with an exit $\phi(\kappa) \in f(\kappa)$ such that $\phi(\kappa) \in f(\lambda_\tau^-) \backslash f(\kappa_\tau^-)$.

(2.5) $\kappa \notin \{\kappa_\tau^-, \lambda_\tau^-\}$, (2.3) and (2.4) are false, and κ is connected with an exit $\phi(\kappa) \in f(\kappa)$ such that $\phi(\kappa) \in \{1, \ldots, k^+\} \backslash (f(\kappa_\tau^-) \cup f(\lambda_\tau^-))$.

For example, the incoming arcs in *Figure 5* are enumerated such that each entrance κ has property $(2.\kappa)$, $\kappa = 1, \ldots, 5$. We can choose $\phi(1) := 1$, $\phi(2) := 7$, $\phi(3) := 2$, $\phi(4) := 7$ and $\phi(5) := 1$..

It is clear that (2.1) – (2.5) are disjoint, and we must show that at least one of these assertions is true for every κ. If $\kappa = \kappa_\tau^-$ and $\kappa = \lambda_\tau^-$ then (2.1) and (2.2) follows by (1), resp. The only critical case we must consider is that $\kappa \notin \{\kappa_\tau^-, \lambda_\tau^-\}$ and (2.3), (2,4) are false. Then negation of (2.3) and (2.4) implies that

$(\diamond)$ $\quad f(\kappa) \subseteq (f(\kappa_\tau^-) \cap f(\lambda_\tau^-)) \cup (\{1, \ldots, k^+\} \backslash (f(\kappa_\tau^-) \cup f(\lambda_\tau^-)))$.

Moreover,

$(\diamond\diamond)$ $\quad f(\kappa)$ is *not* a subset *of* $(f(\kappa_\tau^-) \cap f(\lambda_\tau^-))$.

This can be seen as follows: First $f(\kappa_\tau^-) \cap f(\lambda_\tau^-)$ is a proper subset of $f(\kappa_\tau^-)$ because $f(\kappa_\tau^-) \neq f(\lambda_\tau^-)$. Hence $f(\kappa) \subsetneq f(\kappa_\tau^-)$ if $(\diamond\diamond)$ were false. But this were a contradiction to the minimality of $f(\kappa_\tau^-)$. — Fact (2) follows from $(\diamond)$ and $(\diamond\diamond)$.

We next construct the τ-graph $\mathcal{G}$ if a τ^*-graph $\mathcal{G}$ is given. For this we consider a τ^*-node u^* in $\mathcal{G}$ (see *Figure 6.a)*); its incoming arcs are $r_1^-(u^*), r_2^-(u^*)$, its outgoing arrows are $r_1^+(u^*), r_2^+(u^*)$. Let $\alpha_i(u^*) := \alpha(r_i^-(u^*))$ and $\omega_i(u^*) := \omega(r_i^+(u^*))$, $i = 1, 2$. Without loss of generality we assume that

$(\#)$ $\alpha_i(u^*)$ and $\omega_i(u^*)$ are not valve nodes $(i = 1, 2)$.

Then u^* is replaced by a valve $u := \zeta(u^*)$ of type τ; this is illustrated in *Figure 6.b)*; in particular, we generate the following arcs:

. For all $\kappa \in \{1, \ldots, k^-\} \backslash \{\kappa_\tau^-, \lambda_\tau^-\}$ draw an arc from s to u ending at entrance κ.
For $\kappa = \kappa_\tau^-$ $[\kappa = \lambda_\tau^-$, resp.$]$ replace $r_1^-(u^*)$ $[r_2^-(u^*)]$ by an arc from $\alpha_1(u^*)$ $[\alpha_2(u^*)]$ to the κth entrance of u.

For all $\kappa \in f(\kappa_\tau^-) \cap f(\lambda_\tau^-)$ draw an arc from the κth exit of u to a dead end t_u.
For all $\kappa \in f(\kappa_\tau^-) \backslash f(\lambda_\tau^-)$ $[\kappa \in f(\lambda_\tau^-) \backslash f(\kappa_\tau^-)$, resp.$]$ draw an arc from the κth exit of u to $\omega_1(u^*)$ $[\omega_2(u^*)]$; these arcs replace $r_1^+(u^*)$ $[r_2^+(u^*)$, resp.$]$.
For all remaining $\kappa \in \{1, \ldots, k^+\}$ draw a direct connection from the κth exit of u to the end node t.

This construction may yield parallel arcs; they can be avoided by replacing each of them by a path of length two (see *Figure 6.c)*). So for all $\kappa^- \in \{1, \dots, k^-\}$ the following is true: If $\kappa^- = \kappa_\tau^-$, $\kappa^- = \lambda_\tau^-$ and $\kappa^- \notin \{\kappa_\tau^-, \lambda_\tau^-\}$ there exists a path starting from $\alpha_1(u^*)$, $\alpha_2(u^*)$ and s, resp. that enters u in its κ^-th entrance. This path is called $\langle\alpha_1(u^*), u, \kappa^-\rangle$, $\langle\alpha_2(u^*), u, \kappa^-\rangle$, $\langle s, u, \kappa^-\rangle$, resp. and has no valves up to u. Moreover, for all $\kappa^+ \in \{1, \dots, k^+\}$ there exists a path $\langle\kappa^+, u, w\rangle$ starting from the κ^+th exit of u and ending in $w = \omega_1(u^*)$, $w = \omega_2(u^*)$, $w = t$ and $w = t_u$ if $\kappa^+ \in f(\kappa_\tau^-)\backslash f(\lambda_\tau^-)$, $\kappa^+ \in f(\lambda_\tau^-)\backslash f(\kappa_\tau^-)$, $\kappa^+ \notin f(\kappa_\tau^-) \cup f(\lambda_\tau^-)$ and $\kappa^+ \in f(\kappa_\tau^-) \cap f(\lambda_\tau^-)$, resp.

Substituting each node u^* in $\mathcal{G}^*$ we obtain a τ-graph $\mathcal{G}$. For all τ-nodes u we have the functions $\eta : \mathcal{R}^-(u) \to \{1, \dots, k^-\}$ and $\chi : \mathcal{R}^-(u) \to \{1, \dots, k^+\}$ mentioned in *Definition 3.2.c)*. Moreover, the τ-valve u and the corresponding τ^*-valve u^* have the following properties, which are based on (2.1) – (2.5):

(3) Given an adjustment $\vartheta(u)$ and let $\kappa^- := \eta(\vartheta(u))$; $\phi(\kappa^-)$ is defined as in (2.1) – (2.5).

(3.1) If (2.1) is true for κ^- (i.e. $\kappa^- = \kappa_\tau^-$) then ϑ connects $\langle\alpha_1(u^*), u, \kappa^-\rangle$ with $\langle\phi(\kappa^-), u, \omega_1(u^*)\rangle$.

(3.2) If (2.2) is true for κ^- (i.e. $\kappa^- = \lambda_\tau^-$) then ϑ connects $\langle\alpha_2(u^*), u, \kappa^-\rangle$ with $\langle\phi(\kappa^-), u, \omega_2(u^*)\rangle$.

(3.3) If (2.3) is true for κ^- then ϑ connects $\langle s, u, \kappa^-\rangle$ with $\langle\phi(\kappa^-), u, \omega_1(u^*)\rangle$.

(3.4) If (2.4) is true for κ^- then ϑ connects $\langle s, u, \kappa^-\rangle$ with $\langle\phi(\kappa^-), u, \omega_2(u^*)\rangle$.

(3.5) If (2.5) is true for κ^- then ϑ connects $\langle s, u, \kappa^-\rangle$ with $\langle\phi(\kappa^-), u, t\rangle$.

Figure 6

We next show the equivalence of the following assertions:

(A) All valve adjustments in $\mathcal{G}$ allow an s-t-path.

(B) All valve adjustments in $\mathcal{G}^*$ allow an s-t-path.

Proof of (A) $\Rightarrow$ (B) : Given a valve adjustment ϑ^* in $\mathcal{G}^*$. We must show the existence of a ϑ^*-admissible s-t-path P^* in this graph.

For this we define a valve adjustment ϑ in $\mathcal{G}$: Let u be a valve of $\mathcal{G}$ and $u^* := \zeta^{-1}(u)$. If $\vartheta^*(u^*) = r_1^-(u^*)$ then choose the κ_τ^-th entrance of u, i.e. $\vartheta(u) := \eta^{-1}(\kappa_\tau^-)$; if $\vartheta^*(u^*) = r_2^-(u^*)$ then $\vartheta(u) := \eta^{-1}(\lambda_\tau^-)$.

Then (A) yields a ϑ-admissible s-t-path P. We assume that P visits the valve nodes $u_1, \ldots, u_m$ in this order and define $u_\mu^* := \zeta^{-1}(u_\mu)$, $\mu = 1, \ldots, m$. Then for all μ the following is true:

IF $\vartheta^*(u_\mu^*) = r_1^-(u_\mu^*)$ then the definition of ϑ implies that the κ_τ^-th entrance of u_μ is connected with all exits in $f(\kappa_\tau^-)$. Note that P does *not* use an exit $\kappa \in f(\kappa_\tau^-) \cap f(\lambda_\tau^-)$, which would direct P to the wrong end node t_{u_μ}. Consequently, P leaves u_μ via an exit $\kappa^+ \in f(\kappa_\tau^-) \backslash f(\lambda_\tau^-)$. But this means that P uses the path $\langle \kappa^+, u_\mu, \omega_1(u_\mu^*) \rangle$ after entering u_μ via $\langle \alpha_1(u_\mu^*), u_\mu, \kappa_\tau^- \rangle$. Consequently, $Q_\mu := \langle \alpha_1(u_\mu^*), u_\mu, \kappa_\tau^- \rangle \oplus \langle \kappa^+, u_\mu, \omega_1(u_\mu^*) \rangle$ is a ϑ-admissible subpath of P from $\alpha_1(u_\mu^*)$ to $\omega_1(u_\mu^*)$, while the path $Q_\mu^* := r_1^-(u_\mu^*) \oplus r_1^+(u_\mu^*)$ is a ϑ^*-admissible $\alpha_1(u_\mu^*)$-$\omega_1(u_\mu^*)$-path in $\mathcal{G}^*$.

IF $\vartheta^*(u_\mu^*) = r_2^-(u_\mu^*)$ then an analogous argumentation shows that P has the ϑ-admissible subpath $Q_\mu = \langle \alpha_2(u_\mu^*), u_\mu, \kappa_\tau^- \rangle \oplus \langle \kappa^+, u_\mu, \omega_2(u_\mu^*) \rangle$ from $\alpha_2(u_\mu^*)$ to $\omega_2(u_\mu^*)$, and the path $Q_\mu^* := r_2^-(u_\mu^*) \oplus r_2^+(u_\mu^*)$ is a ϑ^*-admissible $\alpha_2(u_\mu^*)$-$\omega_2(u_\mu^*)$ path in $\mathcal{G}^*$.

Figure 7

So we can write P as $A_1 \oplus Q_1 \oplus A_2 \oplus Q_2 \oplus \cdots \oplus Q_m \oplus A_{m+1}$ where $A_1, \ldots, A_{m+1}$ are free of valves. Then $P^* := A_1 \oplus Q_1^* \oplus A_2 \oplus Q_2^* \oplus \cdots \oplus Q_m^* \oplus A_{m+1}$ is a ϑ^*-admissible s-t-path in $\mathcal{G}^*$

Proof of $(B) \Rightarrow (A)$: Given a valve adjustment ϑ in $\mathcal{G}$. We must prove the existence of an s-t-path P in this graph.
This is true if there exists a valve u such that (2.5) is true for $\kappa^- := \eta(\vartheta(u))$. In this case, (3.5) yields the path $P := \langle s, u, \kappa^- \rangle \oplus \langle \phi(\kappa^-), u, t \rangle$.
Otherwise all valves u have the property that one of the assertions (2.1) – (2.4) is true for $\eta(\vartheta(u))$. Then we define the following adjustment ϑ^* in $\mathcal{G}^*$: Given a valve u^* in $\mathcal{G}^*$ and let $u := \zeta(u^*)$. If (2.1) or (2.3) is true for $\eta(\vartheta(u))$ then $\vartheta^*(u^*) := r_1^-(u^*)$, else $\vartheta^*(u^*) := r_2^-(u^*)$.
Assertion (B) yields a ϑ^*-admissible s-t-path

$$Q^* := [u_0^* = s, u_1^*, u_2^*, \ldots, u_M^* = t] = r_1 \oplus \ldots \oplus r_M \quad \text{where} \quad r_\mu = (u_{\mu-1}^*, u_\mu^*),\ \mu = 1, \ldots, M.$$

We assume that $u_{i(\mu)}^*$, $\mu = 1, \ldots, m$ are the valves on this path where $i(\mu) \le i(\mu+1) - 2$ for all μ (see (#)). An example with $m = 3$ can be found in *Figure 7a)*. Let $u_{i(\mu)} := \zeta(u_{i(\mu)}^*)$, $\mu = 1, \ldots, m$. We consider two cases:

CASE 1: Every entrance $\eta(\vartheta(u_{i(\mu)}))$, $\mu = 1, \ldots, m$, satisfies condition (2.1) or (2.2), i.e. $\eta(\vartheta(u_{i(\mu)})) \in \{\kappa_\tau^-, \lambda_\tau^-\}$
Then P is constructed as shown in *Figure 7.b)*: For all μ the following is true:
IF $\eta(\vartheta(u_{i(\mu)})) = \kappa_\tau^-$ then $\vartheta^*(u_{i(\mu)}^*) = r_1^-(u_{i(\mu)}^*)$ because of the definition of ϑ^*. Hence the subpath $r_{i(\mu)} \oplus r_{i(\mu)+1} = r_1^-(u_{i(\mu)}^*) \oplus r_1^+(u_{i(\mu)}^*)$ connects $\alpha_1(u_{i(\mu)}^*)$ with $\omega_1(u_{i(\mu)}^*)$ (and not $\alpha_2(u_{i(\mu)}^*)$ with $\omega_2(u_{i(\mu)}^*)$). On the other hand, consider $\mathcal{G}$: The assumption $\eta(\vartheta(u_{i(\mu)})) = \kappa_\tau^-$ and (3.1) imply that ϑ connects $\left\langle \alpha_1(u_{i(\mu)}^*), u_{i(\mu)}, \kappa_\tau^- \right\rangle$ with $\left\langle \phi(\kappa_\tau^-), u_{i(\mu)}, \omega_1(u_{i(\mu)}^*) \right\rangle$; consequently, $S_\mu := \left\langle \alpha_1(u_{i(\mu)}^*), u_{i(\mu)}, \kappa_\tau^- \right\rangle \oplus \left\langle \phi(\kappa_\tau^-), u_{i(\mu)}, \omega_1(u_{i(\mu)}^*) \right\rangle$ is a ϑ-admissible connection from $\alpha_1(u_{i(\mu)}^*)$ to $\omega_1(u_{i(\mu)}^*)$ via $u_{i(\mu)}$. This means that the current valve adjustment allows a path from $\alpha_1(u_{i(\rho)}^*)$ to $\omega_1(u_{i(\rho)}^*)$ in *both* graphs $\mathcal{G}^*$ and $\mathcal{G}$.
IF $\eta(\vartheta(u_{i(\mu)})) = \lambda_\tau^-$ then an analogous argumentation yields that $r_{i(\mu)} \oplus r_{i(\mu)+1} = r_2^-(u_{i(\mu)}^*) \oplus r_2^+(u_{i(\mu)}^*)$ is an admissible connection from $\alpha_2(u_{i(\mu)}^*)$ to $\omega_2(u_{i(\mu)}^*)$ in $\mathcal{G}^*$, and the same is true for the path $S_\mu := \left\langle \alpha_1(u_{i(\mu)}^*), u_{i(\mu)}, \lambda_\tau^- \right\rangle \oplus \left\langle \phi(\lambda_\tau^-), u_{i(\mu)}, \omega_2(u_{i(\mu)}^*) \right\rangle$ in $\mathcal{G}$.
So we can transform $Q^* = r_1 \oplus \cdots \oplus r_M$ into the following ϑ-admissible s-t-path in $\mathcal{G}$:

$$P := r_1 \oplus \cdots r_{i(1)-1} \oplus S_1 \oplus r_{i(1)+2} \oplus \cdots \oplus r_{i(2)-1} \oplus S_2 \oplus r_{i(2)+2} \oplus \ldots\ldots \oplus r_{i(m)-1} \oplus S_m \oplus r_{i(m)+2} \oplus \cdots \oplus r_M .$$

There exists a μ' such that $\eta(\vartheta(u_{\mu'}))$ has property (2.3) or (2.4).
We assume that μ'' is the maximum of these numbers μ'; e.g., $\mu'' = 2$ in *Figure 7.c)*. Let $\widetilde{\kappa} := \eta(\vartheta(u_{i(\mu'')}))$.
IF $\widetilde{\kappa}$ has property (2.3) then the definition of ϑ^* implies that in $\mathcal{G}^*$ the arc $r_1^+(u_{i(\mu'')}^*)$ is opened so that $\widetilde{s} := \omega_1(u_{i(\mu'')}^*) = u_{i(\mu'')+1}^*$. On the other hand consider $\mathcal{G}$ and recall (3.3); so we obtain a ϑ-admissible path

$$Q^\circ := \left\langle s, u_{i(\mu'')}, \widetilde{\kappa} \right\rangle \oplus \left\langle \phi(\widetilde{\kappa}), u_{i(\mu'')}, \omega_1(u_{i(\mu'')}^*) \right\rangle ,$$

which is a path from s to $\omega_1(u_{i(\mu'')}^*) = \widetilde{s}$. This means that both in $\mathcal{G}^*$ and in $\mathcal{G}$ there exists a s-$\widetilde{s}$-path which is admissible under the current valve adjustment.
IF $\widetilde{\kappa}$ has property (2.4) then an analogous argumentation says that Q^* visits $\widetilde{s} := \omega_2(u_{i(\mu'')}^*) = u_{i(\mu'')+1}^*$. When considering $\mathcal{G}$ we use (3.4) and can find the ϑ-admissible s-$\widetilde{s}$-path

$$Q^\circ := \left\langle s, u_{i(\mu'')}, \widetilde{\kappa} \right\rangle \oplus \left\langle \phi(\widetilde{\kappa}), u_{i(\mu'')}, \omega_2(u_{i(\mu'')}^*) \right\rangle .$$

Again both valve adjustments ϑ^* and ϑ allow an s-$\widetilde{s}$-path in $\mathcal{G}^*$ and in $\mathcal{G}$, resp.
Consider the final part of Q^* that starts from $\widetilde{s} = u_{i(\mu'')+1}^*$, i.e. $r_{i(\mu'')+2}, \oplus \ldots \oplus r_M,$. This path is according to CASE 1 as μ'' is maximal. So we can replace it by a ϑ-admissible $\widetilde{s}$-t-path $Q^{\circ\circ}$ in $\mathcal{G}$. Then $P := Q^\circ \oplus Q^{\circ\circ}$ is a ϑ-admissible s-t-path.
So we have seen that assertion (A) is true if (B) is given.

We next show that the existence of blocking valve adjustments can be decided in deterministic polynomial time if a $\mathcal{G}$ is a T-graph with $T \subseteq \mathbf{T}^\bullet$.

Theorem 4.5. Let $T \subseteq \mathbf{T}^\bullet$.
Then $\mathbf{PBVA}_d(T)$ can be solved in polynomial time.

Proof: We give an algorithm $\mathcal{A}$ solving our problem in polynomial time. The procedure is similar to a game: The first player adds more and more arcs in order to generate paths for all valve positions. His adversary tries to block all paths by suggesting a bad valve position $\widetilde{\vartheta}$.

Given the graph $\mathcal{H} = (\mathcal{G}, \widehat{\mathcal{V}}, \gamma)$; for any $v \in \mathcal{V}$ let r_v^- be an incoming arc such that the set $\gamma(r_v^-)$ is minimal. Our algorithm is based on the following variables:

end0, end1:	unsuccessful and successful termination of $\mathcal{A}$, resp.
$\mathcal{R}'$:	set of all arcs that have already been found.
$\widetilde{\vartheta}$:	tentative valve adjustment. $\widetilde{\vartheta}$ is pessimistic: It tries to block each valve w by defining $\widetilde{\vartheta}(w) := r$ where r has not yet been found. If, however, $\mathcal{R}^-(w) \subseteq \mathcal{R}'$ then $\widetilde{\vartheta}(w) := r_w^-$ generating the minimal set of outgoing arcs.
OPEN:	set of all possible start nodes for new arcs.
BLOCKED:	set of the valve nodes which have already been found but must not yet be expanded. $v \in$BLOCKED is moved to OPEN if for all valve adjustments, v can be reached via each incoming arc.
CLOSED:	set of nodes for that all possible outgoing arcs have been generated.

Let us now consider the following algorithm:

Procedure $\mathcal{A}$

1. *Initialization:*
   CLOSED := BLOCKED := $\mathcal{R}'$:= $\emptyset$;   OPEN := s;   end0 := end1 := false;
2. while (end0 = false and end1 = false) do
   - 2.1. if OPEN $\neq \emptyset$ then
     - 2.1.1. Choose an arc $r \in \mathcal{R}\backslash\mathcal{R}'$; with $v := \alpha(r) \in$OPEN;
       if $v \in \widehat{\mathcal{V}}$ then r is even selected from $\gamma(r_v^-)$;
       let $w := \omega(r)$;
     - 2.1.2. $\mathcal{R}' := \mathcal{R}' \cup \{r\}$;
     - 2.1.3. if $(v \notin \widehat{\mathcal{V}}$ and $\mathcal{R}^+(v) \subseteq \mathcal{R}')$ then OPEN := OPEN$\backslash\{v\}$; CLOSED := CLOSED$\cup\{v\}$;
     - 2.1.4. if $(v \in \widehat{\mathcal{V}}$ and $\gamma(r_v^-) \subseteq \mathcal{R}')$ THEN OPEN := OPEN$\backslash\{v\}$; CLOSED := CLOSED$\cup\{v\}$;
     - 2.1.5. if $(w \in \widehat{\mathcal{V}}$ and not $\mathcal{R}^-(v) \subseteq \mathcal{R}')$ then
       BLOCKED := BLOCKED $\cup \{w\}$;
       $\widetilde{\vartheta}(w) := \overline{r}$ where $\overline{r}$ is not yet found (i.e. $\overline{r} \in \mathcal{R}^-(w)\backslash\mathcal{R}'$);
     - 2.1.6. if $\big(w \in \widehat{\mathcal{V}}$ and $\mathcal{R}^-(w) \subseteq \mathcal{R}'$ and $w \notin$CLOSED$\big)$ then
       BLOCKED := BLOCKED $\setminus \{w\}$;
       OPEN := OPEN $\cup \{w\}$;
       $\widetilde{\vartheta}(w) := r_w$;
     - 2.1.7. if $\big(w \notin \widehat{\mathcal{V}}$ and $w \neq t$ and $w \notin$CLOSED$\big)$ then OPEN := OPEN$\cup\{w\}$;
     - 2.1.8. if $(w = t)$ then end1 := TRUE
   - 2.2. if OPEN = $\emptyset$ then end0 := TRUE.
3. IF end0 = TRUE there exists a valve adjustment blocking all s-t-paths;
   IF end1 = TRUE then all valve adjustments allow an s-t-path.

We next prove the correctness and start with the following assertion:

(1) If end0 is true then there exists a valve adjustment ϑ_0 such that no s-t-path is possible.

For this we consider the adjustments $\widetilde{\vartheta}(v)$ found by Procedure $\mathcal{A}$ at the moment of its termination. We define ϑ_0 such that $\vartheta_0(v) = \widetilde{\vartheta}(v)$ for all nodes v that are already known by $\mathcal{A}$.
Then we assume and refute the existence of a ϑ_0-admissible s-t-path $P = r_1 \oplus \cdots \oplus r_k$ in $\mathcal{G}$: Note that t has not yet been found by $\mathcal{A}$ as otherwise end1 and not end0 were true. So $r_k \notin \mathcal{R}'$, and there is a first arrow r_j that is not in $\mathcal{R}'$ when $\mathcal{A}$ terminates.
We next see that $u := \alpha(r_j) \in$ OPEN: If u is not a valve then it has not yet been moved to CLOSED because the condition $\mathcal{R}^-(u) \subseteq \mathcal{R}'$ of Step 2.1.3 is violated by $r_j \in \mathcal{R}^-(u) \backslash \mathcal{R}'$. The other case is that $u \in \widehat{\mathcal{V}}$. Then the ϑ_0-admissibility of P yields

(1.1) $\vartheta_0(u) = r_{j-1}$

The minimality of j implies that $\widetilde{\vartheta}(u) = \vartheta_0(u) \stackrel{(1.1)}{=} r_{j-1} \in \mathcal{R}'$. But $\widetilde{\vartheta}(u) \in \mathcal{R}'$ is only possible if u has been moved from BLOCKED to OPEN as described in Step 2.1.5 and 2.1.6. To see that u is not yet in CLOSED we observe that $r_{j-1} \stackrel{(1.1)}{=} \vartheta_0(u) = \widetilde{\vartheta}(u) = r_u^-$ where the last equality is effected by the definition of $\widetilde{\vartheta}(u)$ in Step 2.1.6. The admissibility of P implies that $r_j \in \gamma(r_{j-1}) = \gamma(r_u^-)$; moreover, $r_j \notin \mathcal{R}'$ so that $\gamma(r_u^-)$ is not yet a subset of $\mathcal{R}'$; this means that Step 2.1.4 was not yet applied to $v := u$, and u is still in OPEN.
We have seen that $u \in$ OPEN, no matter whether u is a valve or not. But this is a contradition to the fact that end0 = TRUE, which can only happen if OPEN is empty.
Consequently, there is *no* $\vartheta 0$-admissible s-t-path P.

We next show the correctness of Procedure $\mathcal{A}$ if all valve adjustments allow an s-t-path.

(2) If Procedure $\mathcal{A}$ ends with end1 then every valve adjustment ϑ of $\mathcal{G}$ allows an s-t-path.

For this we consider the situation immediately before the J^{th} iteration of Step 2 where $J = 1, 2, 3, \ldots$. Then for all J the following is true:

(2.1) Let $\mathcal{V}' :=$ OPEN $\cup$ BLOCKED $\cup$ CLOSED and $\mathcal{R}'$ be the set of nodes and arcs found at time J, resp.
For all $w \in \mathcal{V}' \backslash \widehat{\mathcal{V}}$ and all valve adjustments ϑ there exists a ϑ-admissible and elementary s-w-path P_ϑ, which only uses arcs of $\mathcal{R}'$.

(2.2) If, however, $w \in \mathcal{V}'$ is a valve node then we can also find a path P_ϑ as described above; moreover, we even can prescribe the arc $r \in \mathcal{R}^-(w) \cap \mathcal{R}'$ that P_ϑ uses to enter w.

Proof to (2.1) and (2.2): We prove these assertions by an induction on J. If $J = 1$ we have the situation immediately after the initialization; in particular, $\mathcal{V}' = \{s\}$, and (2.1), (2.2) are trivial ($w = s$).
We assume that (2.1) and (2.2) are true for J; we only have to consider the critical case that $\mathcal{V}'$ is changed in the J^{th} iteration, i.e., a new node w is found, which is the endpoint of the current arc $r = (v, w)$.
Given a valve adjustment ϑ. Then the assumption of our induction yields an elementary and ϑ-admissible s-v-path Q_ϑ consisting only of arcs in $\mathcal{R}' \backslash \{r\}$. We next try and connect Q_ϑ with r where the connection at v must accord to ϑ. If v is <u>not</u> a valve then $Q'_\vartheta := Q_\vartheta \oplus r$ is automatically ϑ-admissible. If, however, $v \in \widehat{\mathcal{V}}$ then the choice of r in the current iteration implies that $v \in$ OPEN and $\mathcal{R}^-(v) \subseteq \mathcal{R}'$. Applying (2.2) to v says that Q_ϑ can even be required to enter v via the particular arc $r^- := \vartheta(v) \in \mathcal{R}^-(v) = \mathcal{R}^-(v) \cap \mathcal{R}'$. Then $r \stackrel{Step\ 2.1.1}{\in} \gamma(r_v^-) \stackrel{Lemma\ 4.3}{\subseteq} \gamma(r^-)$. So the path $Q'_\vartheta := Q_\vartheta \oplus r$ does not effect a conflict with ϑ at node v and all earlier nodes.
It remains to transform Q'_ϑ into an *elementary* s-v-path P_ϑ; for this we distinguish between two cases:
CASE 1: w is not a valve node.
Note that Q_ϑ is elementary. If this is not the case for Q'_ϑ then w must have occurred in Q_ϑ and r visits it for the second time. Then let P_ϑ be the part of Q_ϑ from s to the first occurence of w.
CASE 2: w is a valve node.
Then $P_\vartheta := Q'_\vartheta$ is already elementary. *Otherwise* w occured twice on Q'_ϑ because Q_ϑ is elementary. Hence Q'_ϑ uses an arc r' starting from w. The assumption of our induction yields $r' \in \mathcal{R}'$, i.e., r' has been found at Step 2.1.1 of the J^{th} iteration or earlier. But at this moment, w is still in BLOCKED; the earliest moment when w can be moved to OPEN is Step 2.1.4 of the J^{th} iteration. But the fact that $w \in$ BLOCKED makes it impossible

that r' has already been found, i.e. $r' \notin \mathcal{R}' \cup \{r\}$, a contradiction.

We now have proven the assertions (2.1) and (2.2). Fact (2) follows immedeately by applying (2.1) to the node $w := t$.

So we have shown the correctness of Procedure $\mathcal{A}$. It is easy to see that it works in polynomial time: Every occurence of Step 2.1 generates a new arc; so the algorithm must stop at the latest when all $O(|\mathcal{V}|^2)$ arcs of $\mathcal{G}$ are found. ■

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