Dynamic Programming in a Generalized Decision Model

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Abstract: We present two dynamic programming strategies for a general class of decision processes. Each of these algorithms includes among others the following graph theoretic optimization algorithms as special cases:

- the Ford-Bellman Strategy for optimal paths in acyclic digraphs,
- the Greedy Method for optimal forests and spanning trees in undirected graphs.

In our general decision model, we define several structural properties of cost measures in order to formulate sufficient conditions for the correctness of our algorithms.

Our first algorithm works as fast as the original Ford-Bellman Strategy and the Greedy Method, respectively. Our second algorithm solves a larger class of optimization problems than our first search strategy.

1) Ernst-Moritz-Arndt-Universität Greifswald, Fachrichtungen Mathematik/Informatik,
Jahnstr. 15 a, D-17487 Greifswald, GERMANY

Introduction

Two of the most prominent graph theoretic optimization strategies are the Ford-Bellman Algorithm ([2], [8]) and Kruskal's Greedy Method (see [7]).

The first of them finds cost minimal paths in a digraph G. More precisely, if s is a fixed start node then the Ford-Bellman Strategy finds a cost minimal path $\psi(a)$ from s to a for each vertex a. If G is acyclic then this optimization method can be simplified.

Kruskals Greedy Method, however, outputs a minimal spanning tree of an undirected graph with N nodes; more precisely, the algorithm finds a cost minimal forest with a edges for each $a = 0, \ldots, N-1$.

It turns out that both optimization problems can be formulated in a general and uniform way:

PROBLEM **P**

Given the following objects:

- a search space $\mathcal{U}$.
  (In the first example, define $\mathcal{U}$ as the set of all paths starting at s.
  In the second case, $\mathcal{U}$ denotes the set of all forests in the given undirected graph.)
- a set $\mathcal{A}$.
  (In the first example, $\mathcal{A}$ consists of all nodes in the given digraph.
  In the second case, $\mathcal{A} = \{0, \ldots, N-1\}$.)
- a relation $R \subseteq \mathcal{U} \times \mathcal{A}$.
  (In the first example define R such that $(X, a) \in R$ means that the path X ends at the node a.
  In the second case, we define R as follows: $(X, a) \in R$ iff the forest X has exactly a edges.
  In both examples, R is even a function; $R(X)$ is the end node of X and the cardinality of X, respectively.)
- a real-valued cost measure $C : \mathcal{U} \to \mathbb{R}$.

For all $a \in \mathcal{A}$ find a cost optimal object $\psi(a)$ which is related to a. This means that $C(\psi(a))$ is minimal among all candidates $C(X)$ where $(X, a) \in R$.

This general formulation includes several further optimization problems. Important examples arise from searching optimal sequences of decisions in dynamic programs [9] and in sequential decision processes [5], [6]. Also in these situations, the relation R can be defined as a function. In our paper, however, we carefully study a situation where the relation R is not a function.

The main purpose of this paper is to present an efficient solution to the general optimization problem $\mathbf{P}$. For this we introduce two Dynamic Programming algorithms, which are called DP$_1$ and DP$_2$. Their global structure is the following:

> If $a \in \mathcal{A}$ then $\psi(a)$ is the optimal object within the set $\mathcal{M}$ of candidates; this set $\mathcal{M}$ is constructed by applying elementary operations to optimal objects $\psi(a')$ that have been found earlier.

It turns out that the first algorithm DP$_1$ does the same as the Ford-Bellman Strategy if DP$_1$ is used to construct optimal paths in acyclic digraphs; moreover, DP$_1$ behaves like the Greedy Method if DP$_1$ is used to find optimal spanning trees in undirected graphs. We must only define the "elementary operations to optimal objects" in an appropriate way. The second algorithm DP$_2$ is a variant of DP$_2$ that can solve a larger class of problems than DP$_1$.

Our dynamic programming methods seem to be the first that include the case of a relation R which is not a function. In contrast to the algorithms described in [5] and [6], we require a particular structural property of $\mathcal{A}$, that is the so-called admissible ordering of $\mathcal{A}$. An interesting field of future research would be to improve DP$_1$ and DP$_2$ so that they work correctly without admissible orderings of $\mathcal{A}$; these improved versions of our two algorithms would be at least as general as the search strategies in [5] and [9].

The paradigm of Dynamic Programming helps to reduce the time of computation. Since only the optimal objects $\psi(a')$ are considered, no time is wasted by handling non-optimal objects related to a'. In particular, when using DP$_1$ to search optimal paths and minimal spanning trees, DP$_1$ is as fast as the Ford-Bellman Strategy and the Kruskal Method, respectively.

When proving the correctness of DP$_1$ and DP$_2$ we successfully overcome the following difficulty: The Ford-Bellman Procedure, the Greedy-Algorithm and the usual Dynamic Programming strategies are based on different properties of the cost measure C. More precisely, the correct behavior of the Ford-Bellman Procedure is proven with the order preservance of C as defined in [3]. The Greedy procedure works correctly since the set $\mathcal{U}$ of all forests is structured like a matroid. The correctness of the Dynamic Programming paradigm is usually inferred from Bellman Principles, which are described in [1] and [9]. In our paper we translate order preservance, matroid properties and Bellman conditions from the original situations into the general case described in Problem $\mathbf{P}$. These translated properties help us to formulate sufficient conditions for the correct behaviour of DP$_1$ and DP$_2$. We can then prove that these algorithms can actually simulate the Ford-Bellman Procedure for acyclic graphs, the Greedy Algorithm and the usual Dynamic Programming method.

The paper is structured as follows: *Section 1* presents some basic definitions and illustrates them by several examples. *Section 2* compares several structural properties of general decision models such as order preservance and matroid properties. The algorithms DP$_1$ and DP$_2$ are presented in *Section 3* and *4*, respectively; in particular, their correctness is proven and their complexity is analyzed. *Section 5.* consists of concluding remarks. At last, a structural property of a particular cost measure is proven in the *Appendix*.

1. Basic Definitions and Concepts (with Examples)

1.1. Graph Theoretic Definitions.

A *(di-)graph* is defined as a pair $G = (V, E)$ where V is the set of its *nodes* and $E \subseteq V \times V$ is the set of its *arcs*. Given a path $P = (v_1, \ldots, v_n)$ in G; then an *prefix* and a *suffix* of P is a subpath of P starting with v_1 and ending with v_n, respectively; in particular, P is a prefix and a suffix of itself. Given a further path $Q = (w_1, \ldots, w_m)$ or an arc $r = (w_1, w_2)$ with $w_1 = v_n$; then $P \oplus Q$ and $P \oplus r$ is the path generated by appending Q and r to P, respectively. More formally,

$$P \oplus Q := (v_1, \ldots, v_n = w_1, \ldots, w_m) \quad \text{and} \quad P \oplus r := (v_1, \ldots, v_n = w_1, w_2)\,.$$

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1.2 Definition of Relational Decision Models

In this paper, we consider optimization problems in a general situation; we now describe it exactly:

A *relational decision model (RDM)* is a quintuple $\Xi = (\mathcal{U}, \mathcal{A}, R, C, \mathcal{G}_{\mathcal{U}})$ with the following components: $\mathcal{U}$ and $\mathcal{A}$ are two disjoint, nonempty sets of objects; $\mathcal{U}$ contains "many" objects, which have a complicated structure, while $\mathcal{A}$ consists of "few" simply structured elements. $R \subseteq \mathcal{U} \times \mathcal{A}$ is a relation. $C : \mathcal{U} \to \mathbb{R}$ is a cost function from $\mathcal{U}$ into the set $\mathbb{R}$ of all real numbers.

$\mathcal{G}_{\mathcal{U}} = (\mathcal{U}, \mathcal{E}_{\mathcal{U}})$ is a digraph.

If an RDM $\Xi = (\mathcal{U}, \mathcal{A}, R, C, \mathcal{G}_{\mathcal{U}})$ is given, then we formulate the following optimization problem $\mathbf{P}$:

For each $a \in \mathcal{A}$ find and object $O \in \mathcal{U}$ such that $(O, a) \in R$ and $C(O) = \min\{C(O') \,|\, (O', a) \in R\}$.

The graph $\mathcal{G}_{\mathcal{U}}$ has an important influence upon the order in which our dynamic programming algorithms DP_1 and DP_2 consider the elements of $\mathcal{U}$; the set $\mathcal{E}_{\mathcal{U}}$ of all arcs in $\mathcal{G}_{\mathcal{U}}$ is often defined as follows: If $X, Y, \in \mathcal{U}$ then $(X, Y) \in \mathcal{E}_{\mathcal{U}}$ means that Y can be obtained by executing an elementary extension step to X. For example, if X and Y are paths in a digraph, we insert (X, Y) into $\mathcal{E}_{\mathcal{U}}$ iff the path Y is generated by appending an arc at the end of X; in the case of forests in a graph we insert (X, Y) into $\mathcal{E}_{\mathcal{U}}$ iff the forest Y is obtained by adding an edge to X. It turns out that these graphs $\mathcal{G}_{\mathcal{U}}$ force the procedure DP_1 to emulate the Ford-Bellman Algorithm and the Greedy-Algorithm, respectively.

We introduce several further notations related to RDMs. Given a relational decision model $\Xi = (\mathcal{U}, \mathcal{A}, R, C, \mathcal{G}_{\mathcal{U}})$. For all $O \in \mathcal{U}$, $\mathcal{U}' \subseteq \mathcal{U}$ and $a \in \mathcal{A}$ we define

$$R[O] := \{a' \in \mathcal{A} \,|\, (O, a') \in R\}\,, \qquad R[\mathcal{U}'] := \textstyle\bigcup_{O \in \mathcal{U}'} R[O] \quad \text{and} \quad R^{-1}[a] := \{O' \in \mathcal{U} \,|\, (O', a) \in R\}\,.$$

In particular, if R is a function and $O \in \mathcal{U}$ then $R[O]$ and $R[\{O\}]$ denote the set $\{R(O)\}$.
We say that $O \in \mathcal{U}$ is *optimal (minimal)* for some $a \in \mathcal{A}$ iff $(O, a) \in R$, $O \in \mathcal{U}$ and $C(O) = \min\{C(O') \,|\, O' \in R^{-1}[a] \cap \mathcal{U}\}$. So our optimization problem can be formulated as follows: For all $a \in \mathcal{A}$, we search an optimal object $\psi(a)$.
The set of all successors and predecessors of $O \in \mathcal{U}$ within $\mathcal{G}_{\mathcal{U}}$ is abbreviated as $\mathcal{N}_{\mathcal{U}}(O) := \{O' \in \mathcal{U} \,|\, (O, O') \in \mathcal{E}_{\mathcal{U}}\}$ and $\mathcal{N}_{\mathcal{U}}^{-}(O) := \{O' \in \mathcal{U} \,|\, (O', O) \in \mathcal{E}_{\mathcal{U}}\}$, respectively. If $\mathcal{U}' \subseteq \mathcal{U}$ then $\mathcal{N}_{\mathcal{U}}[\mathcal{U}']$ is defined as $\bigcup_{O \in \mathcal{U}'} \mathcal{N}_{\mathcal{U}}(O)$, and $\mathcal{N}_{\mathcal{U}}^{-}[\mathcal{U}']$ is an abbreviation for the set $\bigcup_{O \in \mathcal{U}'} \mathcal{N}_{\mathcal{U}}^{-}(O)$.

For every RDM Ξ we assume that two fixed start objects $O_* \in \mathcal{U}$ and $a_* \in \mathcal{A}$ are given such that $(O_*, a_*) \in R$ and O_* is even optimal for a_*. (This condition is not very restrictive; this is shown in *Remark 1.15.b)*.)

Moreover, we define the quantity $\mu(O, a')$ where $O \in \mathcal{U}$ and $a' \in \mathcal{A}$; roughly spoken, $\mu(O, a')$ denotes the minimal costs of all successors of O that are related to a'. More precisely,

$$\mu(O, a') \;:=\; \inf\{C(O') \,|\, (O, O') \in \mathcal{E}_{\mathcal{U}}\,,\ (O', a') \in R\} \;=\; \inf\{C(O') \,|\, O' \in \mathcal{N}_{\mathcal{U}}(O) \cap R^{-1}[a']\}\,.$$

(If there exists no $O' \in \mathcal{N}_{\mathcal{U}}(O) \cap R^{-1}[a']$ then $\mu(O, a') := \infty$.)

Remark 1.1. Throughout this paper, we make the following assumptions:

(i) R is *total*, i.e. $R[O] \neq \emptyset$ for all $O \in \mathcal{U}$.

(ii) R is *surjective*, i.e. $R^{-1}[a] \neq \emptyset$ for all $a \in \mathcal{A}$;
moreover, there exists an object $O \in R^{-1}[a]$ that is optimal for a.

(iii) All objects $O \in \mathcal{U}$ can be reached from O_*; this means that there exists a path in $\mathcal{G}_{\mathcal{U}}$ from O_* to O.

(iv) For all $O \in \mathcal{U}$ and $a' \in \mathcal{A}$, the infimum $\mu(O, a')$ is a *minimum* if there exists an O' with $(O, O') \in \mathcal{E}_{\mathcal{U}}$ and $(O', a') \in R$. ■

1.3. Examples of Relational Decision Processes

We next illustrate the previous definitions with five examples. The first is the classical problem of optimal paths in digraphs. The second example is a modification of the first and shows that the relation R need not be a function. The third example is the classical Minimal Spanning Tree Problem. The fourth example is a situation in real life, and the fifth is a decision model which frequently occurs in literature.

Example 1.1. *(Optimal Paths in Acyclic Digraphs)*
Given an acyclic digraph $G = (V, E)$ with start node s. For all nodes $v \in E$ we define the sets $N^-(v)$ of all predecessors and $N(v)$ of all successors of v; more formally, $N^-(v) := \{u \in V \,|\, (u,v) \in E\}$ and $N(v) := \{w \in V \,|\, (v,w) \in E\}$. For all arcs $r = (v,w)$ let $\alpha(r) := v$ and $\omega(r) := w$. If P is a path in a graph we define $\alpha(P)$ and $\omega(P)$ is the first and the last node of P.

We next define an RDM $\Xi = (\mathcal{U}, \mathcal{A}, R, C, \mathcal{G}_\mathcal{U})$, which represents the optimization of paths in G.

Let $\mathcal{U}$ be the set $\mathcal{P}(s)$ of all paths in G starting from s. Let $\mathcal{A} := V$. We choose $a_* := s$ as the start object of $\mathcal{A}$. The start object of $\mathcal{U}$ is chosen as the path O_* consisting of the single node $a_* = s$; more formally, $O_* := (a_*) = (s)$. We next define the relation R, which is a function: If $P \in \mathcal{U}$ then let $R(P) \in \mathcal{A}$ be the *last node* of P.
Moreover, the costs of the path P are given by the value $C(P)$. Here we do not generally assume that $C : \mathcal{P}(s) \to \mathbb{R}$ is additive[1].
We next define $\mathcal{G}_\mathcal{U}$. For this we note that every path $P \in \mathcal{U}$ can naturally be extended by appending an arc $r \in E$ at $\omega(P)$; so the following definition is reasonable: $\mathcal{G}_\mathcal{U} := (\mathcal{U}, \mathcal{E}_\mathcal{U})$ with $\mathcal{E}_\mathcal{U} := \{(P, P') \,|\, (\exists\, r \in E)\ P' = P \oplus r\}$.

It should be mentioned that the graph G and the function C must satisfy two conditions, which follow from *Remark 1.1*:

(i) Every $a \in V$ must be reachable from a_* by a path P. This is a consequence of the general assumption that $R^{-1}[a] \neq \emptyset$.

(ii) If a path P starts and ends at a_* then $C(P)$ must not be greater than $C\big((a_*)\big)$ because $O_* = (a_*)$ was assumed to be *optimal* for a_*.

It is clear that the graph $\mathcal{G}_\mathcal{U}$ forms a tree. If $\mathcal{G}_\mathcal{U}$ represents a set $\mathcal{P}(s)$ as just described, then $\mathcal{G}_\mathcal{U}$ is usually called the *expanded version of* G.

(Another reasonable definition of R is that $R(P)$ is the length of P; this point of view often occurs in Dynamic Programming where decision paths of a particular length are optimized.)

In order to find optimal paths in G, the following simplified Ford-Bellman Algorithm is used:

Procedure FORD-BELLMAN

1. Initialize $\psi(a^{(0)})$ as the path $\left(a^{(0)}\right)$ consisting only of $a^{(0)}$.
   Let $\mathcal{L} := \mathcal{L}_0 := \left\{\psi\left(a^{(0)}\right)\right\}$.

2. (Computation of the optimal path $\psi\left(a^{(k)}\right)$, $k = 1, \ldots, n$ from $a^{(0)}$ to $a^{(k)}$; the paths $\psi\left(a^{(j)}\right)$, $j = 0, \ldots, k-1$ are already given immediately before the k-th step; they can be found in the set $\mathcal{L} = \mathcal{L}_{k-1} = \left\{\psi\left(a^{(0)}\right), \ldots, \psi\left(a^{(k-1)}\right)\right\}$.)

   FOR $k := 1$ TO n DO

   2.1 Construct
   $X := \left\{j \in \{0, \ldots, k-1\} \,\middle|\, a^{(j)} \in N^-\left(a^{(k)}\right)\right\}$.

   2.2 Construct $\mathcal{M} := \left\{\psi\left(a^{(j)}\right) \oplus \left(a^{(j)}, a^{(k)}\right) \,\middle|\, j \in X\right\}$.

   2.3 Choose the path $\psi\left(a^{(k)}\right) \in \mathcal{M}$ such that
   $C\left(\psi(a^{(k)})\right) = \min\{C(P) \,|\, P \in \mathcal{M}\}$.

   2.4 $\mathcal{L} := \mathcal{L} \cup \left\{\psi\left(a^{(k)}\right)\right\}$.

Example 1.2. *(Optimal paths with endpoints in a set of nodes)*
We assume that the graph $G = (V, E)$, the start node s, the set $\mathcal{U} = \mathcal{P}(s)$ and the graph $\mathcal{G}_\mathcal{U}$ are defined as in *Example 1.1*.
Given $m \geq 0$ and the sets $a^{(0)}, \ldots, a^{(m)} \subseteq V$ such that $a^{(0)} = \{s\}$ and

$$(1) \qquad a^{(0)} \cup \ldots \cup a^{(m)} = V\,.$$

[1] C is *additive* iff every arc $h(r)$ of G has costs $h(r) \in \mathbb{R}$ and $C(P) = h(r_1) + \ldots + h(r_k)$ for all paths P consisting of the arcs $r_1, \ldots, r_k$.

Let $\mathcal{A} := \{a^{(0)}, \ldots, a^{(m)}\}$. We then define the relation R: A path $P \in \mathcal{U}$ and a set $a^{(j)}$ are in R if the end point of P lies in $a^{(j)}$; this means that

$$R := \left\{ \left(P, a^{(j)}\right) \,\middle|\, \omega(P) \in a^{(j)} \right\} .$$

Note that the sets $a^{(j)}$ need not be pairwise disjoint; therefore R is not always a function.

In contrast to *Example 1.1* we assume that C is *additive*.

Our optimization problem can be formulated as follows:

For each set $a^{(j)}$ find a cost minimal path among all candidates P with $\alpha(P) = a_*$ and $\omega(P) \in a^{(j)}$.

For example, $a^{(j)}$ can be interpreted as a region in the Euclidian plane, and we look for the best paths ending in the areas $a^{(0)}, \ldots, a^{(m)}$.

Example 1.3. *(Minimal Spanning Tree)*

Let $(V, E) = G$ be a finite connected graph and let $\gamma : E \longrightarrow \mathbb{R}$. We define the following relational decision model $\Xi = (\mathcal{U}, \mathcal{A}, R, C, \mathcal{G}_\mathcal{U})$: First, let

$$\mathcal{U} := \{O \subseteq E \,|\, O \text{ forms a forest}\} .$$

Moreover, let $\mathcal{A} := \{0, \ldots, |E|\}$ and $R(O) := |O|$ for all $O \in \mathcal{U}$. We define $C(O) := \sum_{e \in O} \gamma(e)$. In order to introduce the graph $\mathcal{G}_\mathcal{U} = (\mathcal{U}, \mathcal{E}_\mathcal{U})$, we define

$$\mathcal{E}_\mathcal{U} := \left\{(O, O') \in \mathcal{U}^2 \,\middle|\, O \subseteq O' ,\ |O'| = |O| + 1\right\} .$$

The start objects are $O_* := \emptyset$ and $a_* := 0$.

The graph $\mathcal{G}_\mathcal{U}$ is not always a tree because a forest O can be constructed in different ways so that several paths in $\mathcal{G}_\mathcal{U}$ can end with O; hence $\mathcal{G}_\mathcal{U}$ is not isomorphic to a graph of paths described in *Example 1.1*.

(Here it is also possible to replace R by a non-functional relation:
Let $\widetilde{\mathcal{A}} := E$ and let $\widetilde{R} := \{(O, e) \in \mathcal{U} \times E \,|\, e \in O\}$.)

The usual method to find minimal spanning trees is the Greedy Algorithm. The following formulation is almost identical to Kozen's description [7]:

Procedure KRUSKAL

A) Sort the set E of all edges by weight.

B) For each edge e on the list in order of increasing weight, include e in the intermediate forest if e does not form a cycle with the edges already taken.
Otherwise discard e.

We next give another description of the Greedy Algorithm; this formulation makes it easier to see that this search method is a special case of DP_1.

Procedure GREEDY

1. Define $\psi(0) := \emptyset$.
2. FOR $k = 1, \ldots, n$ DO
   Generate $\psi(k)$ by adding an optimal feasible element of E to $\psi(k-1)$.

These two versions of the Greedy Algorithm are equivalent in the case of optimal forests; *Remark 1.8*, however, gives an example where KRUSKAL and GREEDY behave in a different way.

Example 1.4 *(Financial Planning)*

Given the period of the ten years 1991,...,2000. We assume that a saver has \$ 10,000.– in 1991. He can buy and sell bonds during a period of ten years 1991,...,2000. Every bond is a quadruple $b = (i_b, j_b, \eta_b, \delta_b)$ where

- i_b is the year when the bond b is issued,

- j_b is the year when b is sold,
- the factor η_b and the summand δ_b describe the profit arising from buying the bond b.
  More precisely, let x be the invested amount; then b has the value $\eta_b \cdot x$ caused by interest; moreover, the saver gets a fixed premium δ_b. Consequently, an amount of $\lfloor \eta_b \cdot x + \delta_b \rfloor$ is paid to the saver when selling the bond b.

We assume that if the saver buys a bond b then he invests all money available in the year i_b. Moreover, we assume that the saver owns exactly one bond during the whole period from 1991 to 2000. This means that the saver buys a new bond immediately after selling an old one, and he never owns two different types of bonds $b_1 \neq b_2$ in the same year.

We next describe this situation more formally. If the saver has y dollars in the year i then we symbolize this fact by a pair (i, y). We define the set $\mathcal{W}$ of all possible financial situations:

$$\mathcal{W} \;:=\; \{(i, y) \mid i \in \{1991, \ldots, 2000\} \text{ and } y \in \{1, \ldots, 10,000,000\}\}.$$

Suppose that $(i, y) \in \mathcal{W}$ and the saver buys the bond $b = (i, j, \eta, \delta)$; then the situation after selling b is written as $(i, j) \otimes b$; more precisely, $(i, j) \otimes b := (j, J(y, b))$ where $J(y, b) := \lfloor \eta \cdot y + \delta \rfloor$ is the saver's capital after selling the bond b.

Let us now define a relational decison model $\Xi = (\mathcal{U}, \mathcal{A}, R, C, \mathcal{G}_\mathcal{U})$ representing the saver's financial planning and satisfying the conditions of *Remark 1.1.*

The set $\mathcal{U}$ is defined such that it consists of all financial situations that can actually be achieved by buying and selling bonds. More formally,

$$\mathcal{U} \;:=\; \left\{ (i, y) \in \mathcal{W} \;\middle|\; \begin{array}{cc} (i, y) = (1991\,,\, 10,000) & \text{or} \\ (i, y) = (\cdots((1991\,,\, 10,000) \otimes b_1) \otimes \cdots) \otimes b_r & \text{where} \quad b_1, \ldots, b_r \in \mathcal{B} \end{array} \right\}$$

Moreover, let $\mathcal{A} := \{1991, \ldots, 2000\}$

After this we define the relation R as a function: Let $R(i, y) := i$ for all $(i, y) \in \mathcal{U}$. The cost function C is defined with the help of a profit function H. Let $(i, y) \in \mathcal{U}$; then y has been interpreted as the saver's capital in the year i; so it is natural to define $H(i, y) := y$. Let $C(i, y) := -H(i, y)$ for all $(i, y) \in \mathcal{U}$. Then C is a reasonable cost function because minimizing $C(i, y)$ is equivalent to maximizing the profit $H(i, y)$. Finding the optimal object O for some year $i \in \mathcal{A}$ means to find a policy of investments yielding the maximum capital in the year i.

The fact that the saver possesses \$ 10,000.- in the year 1991 is symbolized by the start objects $a_* := 1991$ and $O_* := (1991\,,\, 10\,000)$; note that O_* is the only object related to a_*.

We next introduce the graph $\mathcal{G}_\mathcal{U} = (\mathcal{U}, \mathcal{E}_\mathcal{U})$. We draw an arc from (i, y) to (j, y') iff the situation (j, y') can be effected by buying a bond. More precisely,

$$\mathcal{E}_\mathcal{U} \;:=\; \{((i, y), (i, y) \otimes b) \mid b = (i_b, j_b, \eta_b, \delta_b) \in \mathcal{B} \text{ and } i_b = i\}.$$

Note that $\mathcal{G}_\mathcal{U}$ is not always a tree because different investment policies can yield the same amount y in the same year i so that different paths in $\mathcal{G}_\mathcal{U}$ have the same end point (i, y). Therefore $\mathcal{G}_\mathcal{U}$ cannot be considered as the expanded version of any graph so that the current situation is different from *Example 1.1.*

At last we define the $\mathcal{B}^+(a)$ as the set of all bonds starting in the year a and $\mathcal{B}^-(a')$ as the set of all bonds ending in the year a'. More formally,

$$\begin{array}{lcl} (\forall\, a \in \mathcal{A}) \quad \mathcal{B}^+(a) & := & \{b = (i, j, \eta, \delta) \in \mathcal{B} \mid i = a\}\;, \\ (\forall\, a' \in \mathcal{A}) \quad \mathcal{B}^-(a') & := & \{b = (i, j, \eta, \delta) \in \mathcal{B} \mid j = a'\}\;. \end{array}$$

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Example 1.5 *(Finite dynamic programs and sequential decision processes)*:

Many existing models of decision processes can be embedded into the concept of RDMs. For example, let $\mathcal{D}$ be a finite dynamic program as defined in [9]. Then $\mathcal{D} = (\Omega, D, t, h)$ where Ω is a finite nonempty state space; D is a finite nonempty set of decisions; $t : A \rightarrow \Omega$, where $A \subseteq \Omega \times D$, is the transition mapping; and $h : \mathbb{R} \times A \rightarrow \mathbb{R}$ is the cost function.

Let $y_0 \in \Omega$ be the initial state. We define D^* as the set of all finite sequences of decisions and $\varepsilon \in D^*$ as the empty sequence. Moreover, we introduce an additional dead-end state $y^\perp \notin \Omega$ describing decisions that leave A. Let $\Omega' := \Omega \cup \{y^\perp\}$. We then define a new transition function $t' : \Omega' \times D \to \Omega'$ and a new cost function $h' : \mathbb{R} \times \Omega' \times D \to \mathbb{R}$ as extensions of t and h, respectively.

$$(\forall\, \xi \in \mathbb{R}, u \in \Omega', d \in D) \quad t'(y,d) := \begin{cases} t(y,d) & \text{if} \quad (y,d) \in A, \\ y^\perp & \text{if} \quad (y,d) \notin A, \end{cases} \qquad h'(x,y,d) := \begin{cases} h(\xi,y,d) & \text{if} \quad (y,d) \in A, \\ 0 & \text{if} \quad (y,d) \notin A. \end{cases}$$

(Perhaps the reader expects a high cost value $h'(\xi, y, d)$ for $(y, d) \notin A$ in order to disqualify such pairs (y, d). These pairs, however, are already disqualified because they yield the state $t'(y,d) = y^\perp$.)

We define the following RDM $(\mathcal{U}, \mathcal{A}, R, C, \mathcal{G}_\mathcal{U})$: Let $\mathcal{U} := D^*$ and $\mathcal{A} := \Omega'$. The relation $R \subseteq \mathcal{U} \times \mathcal{A} = D^* \times \Omega'$ is defined such that $(\delta, y) \in R$ iff the sequence δ of decisions yields the state $y \in \Omega'$; more precisely, R is a function with the following recursive description:

$$R(\varepsilon) := y_0\,, \quad (\forall\, \delta \in D^*,\, d \in D) \;\; R(\delta d) := t'(R(\delta), d)\,.$$

We next define the costs C. For this we note Morin's interpretation of the function h: Given a sequence $\delta \in D^*$ and a symbol $d \in D$; let $y := R(\delta)$ is the state effected by δ. If $(y,d) \in A$ and if ξ are the costs of δ, then $h(\xi, y, d)$ is interpreted as the costs of δd. In this line we define C recursively: If $C(\varepsilon)$ is given, then

$$(\forall\, \delta \in D^*,\, d \in D) \quad C(\delta d) \;:=\; h'(\,C(\delta)\,,\, R(\delta)\,,\, d\,)\,.$$

At last, the graph $\mathcal{G}_\mathcal{U} := (\mathcal{U}, \mathcal{E}_\mathcal{U})$ is defined by $\mathcal{E}_\mathcal{U} := \{(\delta, \delta d) \mid \delta \in D^*,\; d \in D\}\,.$

Also the sequential decision processes in [5] and [6] can be considered as RDMs; the definitions are almost the same as in the case of finite dynamic programs.

1.4. Further Definitions in Context with Relational Decision Models

We next define and discuss several structural properties of relational decision models or their components. In particular, we translate the original graph theoretic definition of order preservance [3] and matroid propertiy [7] into the situation of RDMs.

Let us start with order preservance:

Preliminary Remark 1.2. Given the situation of *Example 1.1*; then the order preservance of C is defined as follows:

If two paths $T_1, T_2 \in \mathcal{U}$ end with $R(T_1) = R(T_2) = t \in \mathcal{A}$ and if $U_i := T_i \oplus (t,u)\;\; (i = 1,2)$ with $(t,u) \in E$ then

$$C(T_1) \le C(T_2) \quad\Longrightarrow\quad C(U_1) \le C(U_2)\,.$$

This definition is equivalent to the original version in [3]. In particular, C is order preserving if C is additive; this follows from the equality $C(P_1 \oplus Q) - C(P_1) = C(P_2 \oplus Q) - C(P_2)$ for all paths $P_1, P_2,\, Q$. ∎

Definition 1.3. Given an RDM $\Xi = (\mathcal{U}, \mathcal{A}, R, C, \mathcal{G}_\mathcal{U})$. We say that Ξ is *order preserving* (with respect to R) iff for all $t, u \in \mathcal{A}$ and all $T_1, T_2 \in R^{-1}[t]$ with $u \in R[\mathcal{N}_\mathcal{U}(T_1)] \cap R[\mathcal{N}_\mathcal{U}(T_2)]$ the following is true:

$$C(T_1) \le C(T_2) \quad\Longrightarrow\quad \mu(T_1, u) \le \mu(T_2, u)\,.$$

∎

We next investigate whether or not order preservance is given in *Example 1.1 – 1.5*.

Remark 1.4.

a) Recall *Example 1.1*. Let C be an order preserving cost measure according to *Remark 1.2*. Then the RDM $\Xi = (\mathcal{U}, \mathcal{A}, R, C, \mathcal{G}_{\mathcal{U}})$ described in *Example 1.1* is order preserving as defined in *1.3*.

This can be seen as follows: Given two paths $T_1, T_2 \in \mathcal{U}$ ending at a and let $a' \in V$. Let $i \in \{1, 2\}$. Then $\mathcal{N}_{\mathcal{U}}(T_i) \cap R^{-1}[a'] = \{T_i \oplus (a, a')\}$ if $(a, a') \in E$, and $\mathcal{N}_{\mathcal{U}}(T_i) \cap R^{-1}[a'] = \emptyset$ iff $(a, a') \notin E$. In the first case, $\mu(T_i, a') = C(T_i \oplus (a, a'))$, and in the second case, $\mu(T_i, a') = \infty$ for all $i = 1, 2$. Now let $C(T_1) \leq C(T_2)$. If $(a, a') \in E$ then the graph theoretical order preservance implies that $C(T_1 \oplus (a, a')) \leq C(T_2 \oplus (a, a'))$ so that $\mu(T_1, a') \leq \mu(T_2, a')$. If $(a, a') \notin E$ then $\mu(T_1, a') \leq \infty = \mu(T_2, a')$.

As mentioned above, all additive cost measures C in *Example 1.1* are trivially order preserving in the graph theoretical sense; consequently, the additivity of C implies the order preservance of the RDM Ξ.

b) We next consider *Example 1.2* and an additive cost function C defined with the arc costs $h(r)$, $r \in R$. Let Ξ be the RDM constructed in the description of *Example 1.2*. Then the following condition is sufficient for the order preservance of Ξ: The minimum of all values $h(v, w)$, $w \in a^{(i_2)}$ is the same for all start nodes $v \in a^{(i_1)}$. More precisely

(+) For all $0 \leq i_1 < i_2 \leq n$ there exists a number $g\left(a^{(i_1)}, a^{(i_2)}\right)$ such that

$$\min_{\substack{(v,w) \in E, \\ w \in a^{(i_2)}}} h(v, w) = g\left(a^{(i_1)}, a^{(i_2)}\right) .$$

To see that (+) implies order preservance, we assume that $C(O_1) \leq C(O_2)$ for two paths O_1, O_2 ending with a node in $a^{(i_1)}$. Then $\mu\left(O_\lambda, a^{(i_2)}\right) \overset{(+)}{=} C(O_\lambda) + g\left(a^{(i_1)}, a^{(i_2)}\right)$ $(\lambda = 1, 2)$ so that $\mu\left(O_1, a^{(i_2)}\right) \leq \mu\left(O_2, a^{(i_2)}\right)$.

We next describe how to construct cost measures with property (+). Assume that the sets $a^{(i)}$ are pairwise disjoint. Choose the numbers $g\left(a^{(i_1)}, a^{(i_2)}\right)$ and the functions $\varphi^{(i_1,i_2)} : a^{(i_1)} \to a^{(i_2)}$ where $0 \leq i_1 < i_2 \leq n$. Then define $h : R \to \mathbb{R}$ such that

$$\left(\forall\, i_1, i_2,\ v \in a^{(i_1)},\ w \in a^{(i_2)}\right) \quad \left(\begin{array}{lll} h(v, w) = g\left(a^{(i_1)}, a^{(i_2)}\right) & \underline{\text{if}} & w = \varphi^{(i_1,i_2)}(v)\,, \\ h(v, w) \geq g\left(a^{(i_1)}, a^{(i_2)}\right) & \underline{\text{if}} & w \neq \varphi^{(i_1,i_2)}(v) \end{array} \right) .$$

The pairwise disjointness of the sets $a^{(i)}$ guarantees that no ambivalent definition $h(v, w) := g\left(a^{(i_1)}, a^{(i_2)}\right)$ and $h(v, w) := g\left(a^{(i_1)}, a^{(i_2')}\right)$ with $i_2 \neq i_2'$ can occur because the case $\varphi^{(i_1,i_2)}(v) = w = \varphi^{(i_1,i_2')}(v)$ is impossible.

c) The relational decision model of *Example 1.3* is not always order preserving. A counterexample is given in *Remark 2.5*.

d) The RDM $\Xi = (\mathcal{U}, \mathcal{A}, R, C, \mathcal{G}_{\mathcal{U}})$ of *Example 1.4* is order preserving. To prove this we first observe that for all $a \in \mathcal{A}$ and $O \in \mathcal{U}$ with $R(O) = a$ the following is true:

$$\mathcal{N}_{\mathcal{U}}(O) \;=\; \{O \otimes b \,|\, b \text{ starts in the year } a\} \;= \{O \otimes b \,|\, b \in \mathcal{B}^+(a)\}\,.$$

Consequently

$$(1) \qquad (\forall\, a, a' \in \mathcal{A}\,,\ O \in R^{-1}[a]) \;\; \mathcal{N}_{\mathcal{U}}(O) \cap R^{-1}[a'] \;=\; \{O \otimes b \,|\, b \in \mathcal{B}^+(a) \cap \mathcal{B}^-(a')\}\,.$$

We now assume that two financial situations $O_1, O_2 \in \mathcal{U}$ are given with $R(O_1) = R(O_2) = a$; moreover, let $a' \in R[\mathcal{N}_{\mathcal{U}}(O_1)] \cap R[\mathcal{N}_{\mathcal{U}}(O_2)]$. Then there exist y_1, y_2 such that $O_i = (a, y_i)$, $i = 1, 2$. Moreover, there exists at least one bond of the form (a, a', η, δ). We assume that $b_i^* = (a, a', \eta_i^*, \delta_i^*)$ is the best bond that can be bought in the situation O_i $(i = 1, 2)$ and ends in the year a'; more precisely,

$$(2) \qquad (\forall\, i = 1, 2) \;\; H(O_i \otimes b_i^*) \;=\; \max\{H(O_i \otimes b) \,|\, b \in \mathcal{B}^+(a) \cap \mathcal{B}^-(a')\}\,.$$

Recall $C = -H$ and fact (1); so we obtain

$$(3) \qquad \begin{array}{ll} (\forall\, i = 1, 2) & C(O_i \otimes b_i^*) \;=\; \min\{C(O_i \otimes b) \,|\, b \in \mathcal{B}^+(a) \cap \mathcal{B}^-(a')\} \\ & \;=\; \min\{C(O) \,|\, O \in \mathcal{N}_{\mathcal{U}}(O_i) \cap R^{-1}[a']\} \;=\; \mu(O_i, a') \end{array}$$

Now let $C(O_1) \le C(O_2)$. Then $H(O_1) \ge H(O_2)$, i.e. $y_1 \ge y_2$. We obtain an inequality in which $(*)$ follows from the optimality of b_1^* and $(**)$ is caused by $y_1 \ge y_2$.

$$(4) \qquad \begin{array}{l} H(O_1 \otimes b_1^*) \overset{(*)}{\ge} H(O_1 \otimes b_2^*) = H((a, y_1) \otimes b_2^*) = \\ \lfloor \eta_2^* \cdot y_1 + \delta_2^* \rfloor \overset{(**)}{\ge} \lfloor \eta_2^* \cdot y_2 + \delta_2^* \rfloor = H((a, y_2) \otimes b_2^*) = H(O_2 \otimes b_2^*) . \end{array}$$

Consequently,

$$\begin{array}{rl} \mu(O_1, a') \overset{(3)}{=} & C(O_1 \oplus b_1^*) = -H(O_1 \oplus b_1^*) \\ \overset{(4)}{\le} & -H(O_2 \oplus b_2^*) = C(O_1 \oplus b_1^*) = \mu(O_2, a') . \end{array}$$

e) We describe a sufficient condition for order preservance in *Example 1.5*. Given a finite dynamic program $\mathcal{D} = (\Omega, D, t, h)$. We make an additional monotonicity assumption (MA) about h; it is equivalent to the conditions formulated in [5],[6] and [9].

(MA) For any $\xi_1, \xi_2 \in \mathbb{R}$, $d \in D^*$ and $y \in \Omega$ with $(y, d) \in A$ the following is true:
$\xi_1 \le \xi_2 \implies h(\xi_1, y, d) \le h(x_2, y, d)$.

Then the corresponding RDM $\Xi = (\mathcal{U}, \mathcal{A}, R, C, \mathcal{G}_{\mathcal{U}})$ is order preserving.

Proof: If (MA) is given then h' satisfies the following condition:

(MA') For any $\xi_1, \xi_2 \in \mathbb{R}$, $d \in D^*$ and $y \in \Omega$,
$\xi_1 \le \xi_2 \implies h'(\xi_1, y, d) \le h'(\xi_2, y, d)$.

The proof of (MA') is easy: If $(y, d) \in A$ then $\xi_1 \le \xi_2$ implies that $h'(\xi_1, y, d) = h(\xi_1, y, d) \overset{(MA)}{\le} h(\xi_2, y, d) = h'(\xi_2, y, d)$; if not $(y, d) \in A$ then $h'(\xi_1, y, d) = 0 = h'(\xi_2, y, d)$.

Now let $\delta_1, \delta_2 \in \mathcal{U} = D^*$ with $C(\delta_1) \le C(\delta_2)$. Let $a = R(\delta_1) = R(\delta_2)$ and let $a' \in \mathcal{A} = \Omega$ with $a' \in R[\mathcal{N}_{\mathcal{U}}(\delta_1)] \cap R[\mathcal{N}_{\mathcal{U}}(\delta_2)]$.

Then there exist two decisions $d_1, d_2 \in D$ such that $R(\delta_i d_i) = a'$, $i = 1, 2$; we assume that d_i^*, $i = 1, 2$ is the optimal decision effecting the state a'; more formally,

$$(5) \qquad C(\delta_i d_i^*) = \mu(\delta_i, a') , \quad i = 1, 2 .$$

We next note that $R(\delta_1 d_2^*) = t'(R(\delta_1), d_2^*) = t'(a, d_2^*) = t'(R(\delta_2), d_2^*) = R(\delta_2 d_2^*) = a'$. This implies that $\delta_1 d_2^*$ is one of the candidates when computing $\mu(\delta_1, a')$. Consequently

$$(6) \qquad \mu(\delta_1, a') \le C(\delta_1 d_2^*) .$$

Recalling the assumption $C(\delta_1) \le C(\delta_2)$, we complete the proof of order preservance.

$$\begin{array}{l} \mu(\delta_1, a') \overset{(6)}{\le} C(\delta_1 d_2^*) = h'(C(\delta_1), a, d_2^*) \overset{(MA')}{\le} \\ h'(C(\delta_2), a, d_2^*) = C(\delta_2 d_2^*) \overset{(5)}{=} \mu(\delta_2, a') . \end{array}$$

■

We next generalize the *matroid property*; for this purpose we first introduce the order equivalence of two arcs, and then we describe the transformation of the matroid property from the original situation into the setting of RDMs.

Definition 1.5. Given a relational decision model $\Xi = (\mathcal{U}, \mathcal{A}, R, C, \mathcal{G}_{\mathcal{U}})$. Two arcs $e_i := (U_i, V_i) \in \mathcal{E}_{\mathcal{U}}$ $(i = 1, 2)$ are *order equivalent* $(e_1 \sim e_2)$ iff

$$\begin{array}{l} C(U_1) \le C(U_2) \Longrightarrow C(V_1) \le C(V_2) \quad \text{and} \\ C(U_2) \le C(U_1) \Longrightarrow C(V_2) \le C(V_1) . \end{array}$$

In particular, if $e_1 \sim e_2$ then $C(U_1) = C(U_2) \Rightarrow C(V_1) = C(V_2)$, but the direction " $\Leftarrow$ " is not always true. ■

Preliminary Remark 1.6. We describe two versions of the matroid property and start with the original one:

Original matroid property of the set $\mathcal{U}$ of all forests in graphs:

For all forests V_1 and W in $\mathcal{U}$ and for all numbers a and a', the following is true:

If

$$(+) \quad a \;=\; |V_1| \;<\; |W| \;=\; a'$$

then the following facts are given:

a) There exists a forest $Z \in \mathcal{U}$ of the form

   a1) $V_1 \cup \{\eta\}$ where $\eta \in W \backslash V_1$.

b) Every subset $V_2 \subseteq W$ is an element of $\mathcal{U}$.

We next present a weaker version of the matroid property. The modified condition $(+)$ says that $a' = a - 1$, which is a stronger condition than the above variant of $(+)$; moreover, the modified assertion b) says that only a particular set V_2 must be a subset of W (and not all sets).
The new assertion a2) follows immediately from a1), and b1) – b3) are trivial consequences of the choice of $V_2 := W \backslash \{\eta\}$.
The great advantage of the modified matroid property is that it can be easily translated into the terminology of RDMs.

Weak matroid property of the set $\mathcal{U}$ of all forests in graphs:

For all forests V_0, V_1 and W in $\mathcal{U}$ and for all numbers a and a', the following is true:

If

$$(+) \quad \begin{aligned} V_0 \;&=\; W \backslash \{e\} \;\text{ where }\; e \in W \\ a \;&=\; |V_0| \;=\; |V_1| \quad (=\; |W| - 1)\,, \\ a' \;&=\; |W|\,. \end{aligned}$$

then the following facts are given:

a) There exists a forest $Z \in \mathcal{U}$ of the form

   a1) $Z := V_1 \cup \{\eta\}$ where $\eta \in W \backslash V_1$.

   The forest Z has the property that

   a2) $a' \;=\; |Z| \;\; (= |W| = |V_1| + 1)\,.$

b) There exists a $V_2 \in \mathcal{U}$ (namely $V_2 := W \backslash \{\eta\}$) with the following properties:

   b1) $|V_2| \;=\; a \quad (=\; |V_1|)\,.$

   b2) W can be obtained by adding a single edge to V_2. (This edge is η.)

   b3) If $\;C(V_1) \left\{ \begin{matrix} \leq \\ \geq \end{matrix} \right\} C(V_2)\;$ then

$$C(Z) \;=\; C(V_1) + \gamma(\eta) \;\left\{ \begin{matrix} \leq \\ \geq \end{matrix} \right\}\; C(V_2) + \gamma(\eta) \;=\; C(W)\,.$$

■

The second version of the matroid property is really weaker than the first. This can be seen with the help of the following example: Given an undirected graph $G = (V, E)$ with $E = \{e_0, e_1, e_2\}$. We define

$$\mathcal{U} \;:=\; \left\{ \emptyset\,,\ \{e_0\}\,,\ \{e_0, e_1\}\,,\ \{e_0, e_1, e_2\} \;=E \right\}.$$

Let $\gamma(e_i) := 3 - i\,(i = 0, 1, 2)$ and let $C(O) := \sum_{e \in O} \gamma(e)$ for all $O \in \mathcal{U}$.
Then $\mathcal{U}$ has the weak matroid property. But $\mathcal{U}$ does not have the original matroid property because the subset $V_2 := \{e_1, e_2\}$ of $W' := E$ is not in $\mathcal{U}$. ■

We next translate the above weak matroid property of $\mathcal{U}$ into the situation of relational decision models:

Definition 1.7. Given an RDM $\Xi = (\mathcal{U}, \mathcal{A}, R, C\mathcal{G}_{\mathcal{U}})$. Then Ξ has the *matroid property* iff the following condition is satisfied:

For all objects $V_0, V_1, W \in \mathcal{U}$ and for all elements $a, a' \in \mathcal{A}$, the following is true:

If

$$(+) \qquad \begin{array}{l} (V_0, W) \in \mathcal{E}_{\mathcal{U}}\,, \\ (V_0, a) \in R \ \text{ and } \ (V_1, a) \in R\,, \\ (W, a') \in R \end{array}$$

then the following facts are given:

a) There exists an object $Z \in \mathcal{U}$ with the property

   a1) $Z \in \mathcal{N}_{\mathcal{U}}(V_1)$.

   The Z has1 the property that

   a2) $(Z, a') \in R$.

b) There exists an object $V_2 \in \mathcal{U}$ such that

   b1) $(V_2, a) \in R$,

   b2) $(V_2, W) \in \mathcal{E}_{\mathcal{U}}$

   b3) If $C(V_1) \left\{ \begin{array}{c} \leq \\ \geq \end{array} \right\} C(V_2)$ then

$$C(Z) \left\{ \begin{array}{c} \leq \\ \geq \end{array} \right\} C(W)\,.$$

We now have formulated the matroid property for relational decision models. Before presenting further definitions we study the behavior of the Greedy Algorithm when applied to the example at the end of *Remark 1.6.*

Remark 1.8. We next show that the procedures KRUSKAL and GREEDY given in *Example 1.3* are not equivalent. For this we recall the example for a weak matroid that was not a matroid in the original sense: Given an undirected graph $G = (V, E)$ with $E = \{e_0, e_1, e_2\}$. Let $\mathcal{U} := \left\{ \emptyset\,,\ \{e_0\}\,,\ \{e_0, e_1\}\,,\ \{e_0, e_1, e_2\} = E \right\}$, and let $\mathcal{A} := \{0, 1, 2, 3\}$. For every $O \in \mathcal{U}$ we define $R(O)$ as the cardinality of O. Let $\gamma(e_i) := 3 - i\ (i = 0, 1, 2)$ and let $C(O) := \sum_{e \in O} \gamma(e)$ for all $O \in \mathcal{U}$.

Then our optimization problem has a trivial solution because for each number $a \in \mathcal{A}$ there exists exactly one object $O \in \mathcal{U}$ with $R(O) = |O| = a$.

This optimal solution is actually found by the procedure GREEDY, which constructs

$$\psi(0) = \emptyset\,, \quad \psi(1) = \{e_0\}\,, \quad \psi(2) = \{e_0, e_1\}\,, \quad \psi(3) = \{e_0, e_1, e_2\} \;=\; E\,.$$

The procedure KRUSKAL, however, fails to find the optimal solution because it must discard an unfeasable element $e \in E$ for all following rounds of the algorithm. More precisely:

Let $\psi(k-1) \cup \{e\} \notin \mathcal{U}$ for some k and e; then the element e is not considered any more; that is, KRUSKAL does not construct any set $\psi(k'-1) \cup \{e\}$, $k' > k$

This yields the following behavior of KRUSKAL: When generating $\psi(1)$ the algorithm first tries to add the cheapest element to $\psi(0) = \emptyset$; but $\psi(0) \cup \{e_2\} = \{e_2\}$ is not an element of $\mathcal{U}$. Consequently, KRUSKAL does not find $\psi(3) = \{e_0, e_1, e_2\} = \psi(2) \cup \{e_2\}$ because the discarding rule forbids the construction of the candidate $\psi(k') \cup \{e\}$ with $k' = 2$ and $e = e_2$.

Consequently, KRUSKAL and GREEDY are not equivalent when applied to $\mathcal{U}$. If, however, $\mathcal{U}$ is the set of all forests in a graph then GREEDY automatically complies with the discarding rule; therefore GREEDY generates the same forests as KRUSKAL. ∎

The next introduce the optimal successor property of a relational decision model Ξ. This definition is not directly based on *Examples 1.1* and *1.3*, but it is very useful for proving the correctness of DP_1. Roughly spoken, optimal successor property means the following: If an arbitrary object X_1 has an optimal successor then an optimal object X_2 has an optimal successor, too.

Definition 1.9. Given an RDM $\Xi = (\mathcal{U}, \mathcal{A}, R, C, \mathcal{G}_{\mathcal{U}})$.
We say that Ξ has the *optimal successor property (OSP)* if for all $x, y \in \mathcal{A}$, $X_1, X_2 \in R^{-1}[x]$, $Y_1 \in R^{-1}[y]$ the following is true:

If $(X_1, Y_1) \in \mathcal{E}_{\mathcal{U}}$ and if X_2, Y_1 are optimal for x and y, respectively, then there exists a successor $Y_2 \in \mathcal{N}_{\mathcal{U}}(X_2)$ which is optimal for y. (This implies that $C(Y_2) = C(Y_1)$ but not always $Y_2 = Y_1$.) ∎

Remark 1.10 Given the situation of *Example 1.2*. We construct an additive cost measure C with the property that the relational decision model $(\mathcal{U}, \mathcal{A}, R, C, \mathcal{G}_{\mathcal{U}})$ has the OSP. In contrast to *1.4.b)*, condition (+) need <u>not</u> be satisfied, and the sets $a^{(i)}$ need <u>not</u> be pairwise disjoint.

For this we give the following definitions: If $I \subseteq \mathcal{A} = \{0, \ldots, n\}$ then let $a^{(I)} := \bigcup_{i \in I} a^{(i)}$; in particular, $a^{(\emptyset)} = \emptyset$ and $a^{(\{i\})} = a^{(i)}$ for all i. For all $k = 0, \ldots, n$ we define $\mathcal{X}_k := \left\{ j \in \{0, \ldots, n-1\} \,\middle|\, a^{(j)} \times a^{(k)} \neq \emptyset \right\}$. ($\mathcal{X}_k$ is the same as the set X generated in Step 2.2 of DP_1.) Moreover, let $R > 0$ and $0 < \alpha \leq 2/5$.

We make the following assumptions:

(i) For every $v \in a^{(\mathcal{X}_k)}$ there exists an arc from v into $a^{(k)}$, i.e. $N(v) \cap a^{(k)} \neq \emptyset$.
Moreover, we assume the existence of the minimum $h^*(v)$ of all values $h(v, w)$ where $w \in a^{(k)}$ is a successor of v. We choose the node $w^*(v) \in a^{(k)}$ such that $h(v, w^*(v)) = h^*(v)$.

(ii) For all $v \in a^{(\mathcal{X}_k)}$ and $w \in N(v) \cap a^{(k)}$ with $w \neq w^*(v)$ the following is true: $h(v, w) \geq h(v, w^*(v)) + R \cdot \alpha^{3k}$.

(iii) $\left(\forall\, v \in a^{(\mathcal{X}_k)}\right) \left(\forall w \in a^{(k)},\ w \in N(v)\right) \quad R \cdot \alpha^{3k-1} \leq |h(v, w)| \leq R \cdot \alpha^{3k-2}$.

This cost function C is used to define the RDM $\Xi = (\mathcal{U}, \mathcal{A}, R, C, \mathcal{G}_{\mathcal{U}})$ according to *Example 1.2*; then Ξ has the optimal successor property. The exact proof is given in the appendix. The argumentation is the following: Given the paths $X_1, X_2 \in R^{-1}\left[a^{(j)}\right]$ and $Y_1 \in R^{-1}\left[a^{(k)}\right]$ such that Y_1 is an optimal successor of X_1 and X_2 is optimal for $a^{(j)}$. Then X_1 and X_2 must have the same end point as otherwise Y_1 cannot be optimal. The optimal successor Y_2 of X_2 is constructed by appending the last arc of Y_1 to X_2. ∎

We next define several Bellman properties of relational decsion models. These properties are similar to the principles of optimality in [1] and [9].

Definition 1.11. Given an RDM $\Xi = (\mathcal{U}, \mathcal{A}, R, C, \mathcal{G}_{\mathcal{U}})$.

a) Ξ has the *weak Bellman Property* iff for all $a \in \mathcal{A}$ the following is true:

> There exists an optimal object O for a,
> there exists a path $P = (O_* = O_0, O_1, \ldots, O_r = O)$,
> there exist objects $a_* = a_0, a_1, \ldots, a_r = a$ with $(O_\rho, a_\rho) \in R$, $\rho = 0, \ldots, r$
> such that O_ρ is optimal a_ρ, $\rho = 0, \ldots, r$.

b) Ξ has the *strong Bellman Property* iff for all $a \in \mathcal{A}$ the following is true:

Every optimal objects O for a,
every path $P = (O_* = O_0, O_1, \ldots, O_r = O)$,
all objects $a_* = a_0, a_1, \ldots, a_r = a$ with $(O_\rho, a_\rho) \in R, \ \ \rho = 0, \ldots, r$
have the property that O_ρ is optimal a_ρ, $\rho = 0, \ldots, r$.

c) As an analogy to the weak and the strong Bellman principle we define the $(\mathbf{q}_1\mathbf{q}_2\mathbf{q}_3)$-*Bellman property* where $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3 \in \{\forall, \exists\}$; this Bellman property is given iff

$\big(\mathbf{q}_1\, O \in \mathcal{U}$, O optimal for $a\big)$
$\big(\mathbf{q}_2\, P = (O_* = O_0, O_1, \ldots, O_r = O)\big)$
$\big(\mathbf{q}_3\, a_* = a_0, a_1, \ldots, a_r = a \in \mathcal{A}$ with $(O_\rho, a_\rho) \in R, \ \ \rho = 0, \ldots, r\big)$
O_ρ is optimal a_ρ, $\rho = 0, \ldots, r$.

The weak Bellman principle is equivalent to the ($\exists\exists\exists$)-Bellman property, and the strong Bellman principle is equivalent to the ($\forall\forall\forall$)-Bellman property.

d) We say that a path $P = (O_0 = O_*, \ldots, O_r = O)$ in $\mathcal{G}_\mathcal{U}$ is a *Bellman path* iff there exist $a_* = a_0, \ldots, a_r \in \mathcal{A}$ such that O_j is optimal for a_j for all $j = 1, \ldots, r$.
Let $a \in \mathcal{A}$; then we say that P is a *Bellman path from* (O_*, a_*) *to* (O, a) iff the objects $a_0, \ldots, a_r$ can be chosen such that $a_r = a$. ∎

We discuss these Bellman principles in the next remark:

Remark 1.12.

a) For all $i = 1, 2, 3$ the following is true:

(*) The $(\mathbf{q}_1\mathbf{q}_2\mathbf{q}_3)$-Bellman property becomes weaker if the quantifier $\mathbf{q}_i$ is switched from '$\forall$' to '$\exists$'.

This can be seen as follows: For $i = 1$ note that *Remark 1.1.(ii)* guarantees the existence of an optimal element O for a. For $i = 2$ recall *Remark 1.1.(iii)*, which yields a path P from O_* to O. For $i = 3$ note that *Remark 1.1.(i)* yields an element $a_\rho \in R[O_\rho]$ for every $\rho = 0, \ldots, r$.

b) If the third quantifier is equal to '$\exists$' then every $(\mathbf{q}_1\mathbf{q}_2\mathbf{q}_3)$-Bellman principle can be formulated as follows:

$\big(\mathbf{q}_1\, O \in \mathcal{U}$, O optimal for $a\big)$ $\big(\mathbf{q}_2\, P = (O_* = O_0, O_1, \ldots, O_r = O)\big)$
P is a Bellman path from (O_*, a_*) to (O, a).

c) If R is a function then the $(\mathbf{q}_1\mathbf{q}_2\forall)$- and the $(\mathbf{q}_1\mathbf{q}_2\exists)$-Bellman principles are equivalent. Both properties can be formulated as follows:

$\big(\mathbf{q}_1\, O \in \mathcal{U}$, O optimal for $a = R(O)\big)$ $\big(\mathbf{q}_2\, P = (O_* = O_0, O_1, \ldots, O_r = O)\big)$
O_ρ is optimal $R(a_\rho)$, $\rho = 0, \ldots, r$. ∎

Our next definition introduces two structural properties of the components of relational decision models.

Definition 1.13. Given an RDM $\Xi = (\mathcal{U}, \mathcal{A}, R, C, \mathcal{G}_\mathcal{U})$.

a) R is C-*monotone* iff for all objects $O_1, O_2 \in \mathcal{U}$ related to the same $a \in \mathcal{A}$ the following is true:

$$C(O_1) \le C(O_2) \quad \Longrightarrow \quad R[\mathcal{N}_\mathcal{U}(O_1)] \supseteq R[\mathcal{N}_\mathcal{U}(O_2)]\,.$$

b) The set is $\mathcal{A}$ is *orderable* (with respect to $\mathcal{G}_\mathcal{U}$ and R) iff there exists an enumeration $a_* = a^{(0)}, a^{(1)}, a^{(2)}, \ldots$ of $\mathcal{A}$ with the following property: If (O, O') is an arc of $\mathcal{G}_\mathcal{U}$ and $(O, a^{(i)})$, $(O', a^{(j)}) \in R$ then $i < j$. Such an enumeration is called an *admissible ordering* (with respect to $\mathcal{G}_\mathcal{U}$ and R). ∎

We next make several observations about C-monotonicity.

Remark 1.14.

a) The C-monotonicity of R is automatically given in *Example 1.1, 1.3, 1.4* and *1.5*. The reason is that in each of these examples the set $R[\mathcal{N}_{\mathcal{U}}(O)]$ does not depend on the costs of O.

We next consider the examples in detail: Given $O_1, O_2 \in \mathcal{U}$ and $a \in \mathcal{A}$ as in *Definition 1.13.a)*.

In *Example 1.1* we obtain the following equality:

$$(\forall\, i = 1,2)\;\; R[\mathcal{N}_{\mathcal{U}}(O_i)] \;=\; R[\{O_i \oplus (a,a') \,|\, (a,a') \in E\} \;=\; \{a' \,|\, (a,a') \in E\}$$

so that even $R[\mathcal{N}_{\mathcal{U}}(O_1)] = R[\mathcal{N}_{\mathcal{U}}(O_2)]$.

In *Example 1.3*, the matroid property is given; *Part b)* of this remark says that the matroid property yields the C-monotonicity of R.

In *Example 1.4* we first show

($\bullet$) Given $a, a' \in \mathcal{A}$. Then for all $O \in R^{-1}[a]$ the following is true:
$a' \in R[\mathcal{N}_{\mathcal{U}}(O)] \iff \mathcal{B}^{+}(a) \cap \mathcal{B}^{-}(a') \neq \emptyset$.

To prove $'' \Rightarrow''$ we assume that $a' \in R[\mathcal{N}_{\mathcal{U}}(O)]$; then there exists a bond b with $R(O \otimes b) = a'$. This is only possible if b is of the form $b = (a, a', \eta, \delta)$. The direction $'' \Leftarrow''$ can easily be seen by computing $R(O \otimes b)$ where $b \in \mathcal{B}^{+}(a) \cap \mathcal{B}^{-}(a')$.

The monotonicity of R follows from ($\bullet$): Let $O_1, O_2 \in R^{-1}[a]$ then

$$R[\mathcal{N}_{\mathcal{U}}(O_1)] \overset{(\bullet)}{=} \{a' \in \mathcal{A} \,|\, \mathcal{B}^{+}(a) \cap \mathcal{B}^{-}(a') \neq \emptyset\} \overset{(\bullet)}{=} R[\mathcal{N}_{\mathcal{U}}(O_2)]\,,$$

no matter how the costs $C(O_1)$ and $C(O_2)$ behave.

At last, we consider *Example 1.5*. The assumption that O_1 and O_2 are related to a means that $R(O_1) = R(O_2) = a$. This is used in equality ($\diamond$) of the following assertion: For all $i = 1,2$,

$$R[\mathcal{N}_{\mathcal{U}}(O_i)] \;=\; \{R(O_i d) \,|\, d \in D\} \overset{(\diamond)}{=} \{t'(a,d) \,|\, d \in D\}\,.$$

Hence $R[\mathcal{N}_{\mathcal{U}}(O_1)] \;=\; R[\mathcal{N}_{\mathcal{U}}(O_2)]\,.$

b) Given an RDM $\Xi = (\mathcal{U}, \mathcal{A}, R, C, \mathcal{G}_{\mathcal{U}})$ with the matroid property. Then R is C-monotone.

This can be seen as follows: Let $(O_1, a), (O_2, a) \in R$ and $C(O_1) \leq C(O_2)$. Let $O_2' \in \mathcal{N}_{\mathcal{U}}(O_2)$ and $a' \in R[O_2']$. We must show that then $a' \in R[\mathcal{N}_{\mathcal{U}}(O_1)]$.

For this let $V_0 := O_2$, $V_1 := O_1$ and $W := O_2'$ The matroid property yields an object $Z \in R^{-1}[a']$ and an arc $e_1 = (V_1, Z) \in \mathcal{E}_{\mathcal{U}}$. Hence $Z \in \mathcal{N}_{\mathcal{U}}(V_1)$, and consequently $a' \in R[\mathcal{N}_{\mathcal{U}}(V_1)]$; this and $O_1 = V_1$ imply that indeed $a' \in R[\mathcal{N}_{\mathcal{U}}()_1)]$.

This means that $R[\mathcal{N}_{\mathcal{U}}(O_2)] \supseteq R[\mathcal{N}_{\mathcal{U}}(O_1)]$ and even $R[\mathcal{N}_{\mathcal{U}}(O_2)] = R[\mathcal{N}_{\mathcal{U}}(O_1)]$, because the condition $C(O_1) \leq C(O_2)$ has not actuelly been used. ■

In the next remarks we give examples of admissible orderings; moreover, we show how admissible orderings can be maintained if an artificial start object is added to $\mathcal{U}$.

Remark 1.15.

a) We describe how admissible orderings are realized in our earlier examples:

<u>*Example 1.1.*</u> Since G is acyclic there exists a topological ordering $a^{(\nu)}$, $\nu = 0, \ldots, n$ of $\mathcal{A}$; this means that

(1) $i < j$ for all arcs $\left(a^{(i)}, a^{(j)}\right) \in E$.

This ordering is also admissible in the sense of *Definition 1.13.b)*. To see this we consider two paths $X, Y \in \mathcal{U}$ with $R(X) = a^{(i)}$, $R(Y) = a^{(j)}$ and $(X, Y) \in \mathcal{E}_{\mathcal{U}}$. Then there exists an arc $r \in E$ with $Y = X \oplus r$, and the values $R(X)$, $R(Y)$ say that $r = \left(a^{(i)}, a^{(j)}\right)$. Then $i < j$ by (1).

<u>*Example 1.2.*</u> Here the existence of admissible orderings depends on the particular situation.

Example 1.3. Here the definition $a^{(\nu)} := \nu$ $(\nu = 0, \ldots, n)$ yields an admissible ordering of $\mathcal{A}$. To see this we assume that $(X, Y) \in \mathcal{E}_{\mathcal{U}}$ with $a^{(i)} = R(X)$ and $a^{(j)} = R(Y)$. Then Y is the union of X with a single edge; consequently, $i = a^{(i)} = |X| = |Y| + 1 = a^{(j)} + 1 = j + 1$ so that indeed $i < j$.

Example 1.4. Let $a^{(\nu)} := 1991 + \nu$, $\nu = 0, \ldots, 9$. We show that this ordering is admissible. Let $(X, Y) \in \mathcal{E}_{\mathcal{U}}$, $R(X) = a^{(i)}$ and $R(Y) = a^{(j)}$. Then there exists a bond b with $Y = X \oplus b$, and this bond b must start in the year $a^{(i)}$ and end in the year $a^{(j)}$. Consequently, $i = a^{(i)} - 1991 < a^{(j)} - 1991 = j$.

Example 1.5. Admissible ordering are not automatically given.

b) Adding new start objects $\widetilde{O}_*$ to $\mathcal{U}$ and $\widetilde{a}_*$ to $\mathcal{A}$ does not destroy the existence of an admissible ordering. More precisely, given an RDM $\Xi = (\mathcal{U}, \mathcal{A}, R, C, \mathcal{G}_{\mathcal{U}})$ with the start objects O_* for $\mathcal{U}$ and a_* for $\mathcal{A}$. We assume that $a_* = a^{(0)}, a^{(1)}, \ldots, a^{(n)}$ is an admissible ordering of $\mathcal{A}$.
Then we define a new RDM $\widetilde{\Xi} = (\widetilde{\mathcal{U}}, \widetilde{\mathcal{A}}, \widetilde{R}, \widetilde{C}, \mathcal{G}_{\widetilde{\mathcal{U}}})$ as follows:
Let $\widetilde{\mathcal{U}} := \mathcal{U} \cup \{\widetilde{O}_*\}$ and $\widetilde{\mathcal{A}} := \mathcal{A} \cup \{\widetilde{a}_*\}$ where $\widetilde{O}_* \notin \mathcal{U}$ and $\widetilde{a}_* \notin \mathcal{A}$. Define the relation $\widetilde{R} \subseteq \widetilde{\mathcal{U}} \times \widetilde{\mathcal{A}}$ as $\widetilde{R} := R \cup \{(\widetilde{O}_*, \widetilde{a}_*)\}$. Let $\widetilde{C}(O) := C(O)$ if $O \neq \widetilde{O}_*$ and $\widetilde{C}(\widetilde{O}_*) := 0$. Moreover, construct $\mathcal{G}_{\widetilde{\mathcal{U}}}$ by adding the arc $(\widetilde{O}_*, O_*)$ to $\mathcal{G}_{\mathcal{U}}$. Then the following ordering of $\widetilde{\mathcal{A}}$ is admissible with respect to $\mathcal{G}_{\widetilde{\mathcal{U}}}$ and $\widetilde{R}$: Let $\widetilde{a}^{(0)} := \widetilde{a}_*$, $\widetilde{a}^{(\nu)} := a^{(\nu-1)}$, $\nu = 1, \ldots, n+1$.
Note that the new RDM $\widetilde{\Xi}$ satisfies the condition in *Remark 1.1.(iii)*, since every object $O \in \widetilde{\mathcal{U}}$, $O \neq \widetilde{O}_*$ can be reached from $\widetilde{O}_*$ via O_*.
In addition, the following assertions are true:

($) $\widetilde{R}^{-1}\left[\widetilde{a}^{(0)}\right] = \widetilde{R}^{-1}[\widetilde{a}_*] = \{\widetilde{O}_*\}$.
Hence $\widetilde{O}_*$ is optimal for $\widetilde{a}_* = \widetilde{a}^{(0)}$.

($$) The start object $\widetilde{O}_*$ is not related to any element $\widetilde{a}^{(k)}$, $k > 0$; more formally, $\left(\widetilde{O}_*, \widetilde{a}^{(k)}\right) \notin \widetilde{R}$ for all $k > 0$.

So the conditons ($) and ($$) can be required without loss of generality. ∎

2. Logical Relationships between Different Properties of C

This section consists of proofs or counterexamples to assertions of the form $A \Rightarrow B$ where A and B mean order preservance, optimal successor property, matroid property or some Bellman condition.

We start with the following result:

Theorem 2.1. For all relational decision models $\Xi = (\mathcal{U}, \mathcal{A}, R, C, \mathcal{G}_{\mathcal{U}})$ the following is true:
If Ξ is order preserving and R is C-monotone then Ξ has optimal successor property.

Proof: Let X_1, X_2, Y_1, x, y be the objects described in the definition of OSP. Then the existence of the arc $(X_1, Y_1) \in \mathcal{E}_{\mathcal{U}}$ implies that $y \in R[\mathcal{N}_{\mathcal{U}}(X_1)]$. Hence $y \in R[\mathcal{N}_{\mathcal{U}}(X_2)]$ by C-monotonity.
Therefore we can choose Y_2 such that $C(Y_2) = \mu(X_2, y)$. Then $C(Y_2) = \mu(X_2, y) \leq \mu(X_1, y)$ because $C(X_2) \leq C(X_1)$ and Ξ is order preserving. On the other hand, $\mu(X_1, y) \leq C(Y_1)$ because of the minimality of $\mu(X_1, y)$, and $C(Y_1) \leq C(Y_2)$ as Y_1 is optimal. Consequently, $C(Y_2) \leq \mu(X_1, y) \leq C(Y_1) \leq C(Y_2)$ so that $C(Y_2) = C(Y_1)$ and Y_2 is indeed an optimal successor of X_2 related to y. ∎

Theorem 2.2. For all relational decision models $\Xi = (\mathcal{U}, \mathcal{A}, R, C, \mathcal{G}_{\mathcal{U}})$ the following is true:
If Ξ has matroid property then Ξ has optimal successor property.

Proof: Let X_1, X_2, Y_1, x, y be the objects described in the definition of the OSP.
Then let $V_0 := X_1$, $V_1 := X_2$, $W := Y_1$ and $e_0 := (V_0, W) = (X_1, Y_1) \in \mathcal{E}_{\mathcal{U}}$. Then the matroid property of C implies the existence of two order equivalent arcs $(V_1, Z) = e_1$ and $(V_2, W) = e_2$ with $V_2 \in R^{-1}[x]$ and $Z \in R^{-1}[y]$.
Now note that $C(V_1) \leq C(V_2)$ since $V_1 = X_2$ is optimal. This and $e_1 \sim e_2$ implies that $C(Z) \leq C(W) = C(Y_1)$. Consequently, $C(Z) = C(W) = C(Y_1)$ because $W = Y_1$ has been assumed to be optimal. Hence $Y_2 := Z$ is

optimal, too, and it is a successor of X_2 because $(X_2, Y_2) = (V_1, Z) = e_1 \in \mathcal{E}_\mathcal{U}$ ∎

The next result shows that the OSP can be used to construct paths consisting of optimal nodes.

Lemma 2.3. Given an RDM $\Xi = (\mathcal{U}, \mathcal{A}, R, C, \mathcal{G}_\mathcal{U})$. We assume that Ξ has the optimal successor property.
Let $n \leq 2$. Given the objects $O_{1,1}, \ldots, O_{1,n} \in \mathcal{U}$, $O_{2,1} \in \mathcal{U}$ and $a_1, \ldots, a_n \in \mathcal{A}$ with the following properties:

a) $P_1 := (O_{1,1}, \ldots, O_{1,n})$ is a path in $\mathcal{G}_\mathcal{U}$.

b) $(O_{1,i}\,,\, a_i) \in R\ \ (i = 1, \ldots, n)$, and $(O_{2,1}, a_1) \in R$.

c) $O_{2,1}$ is optimal for a_1, and $O_{1,2}, \ldots, O_{1,n}$ are optimal for $a_2, \ldots, a_n$, respectively.

Then we can find objects $O_{2,2}, \ldots, O_{2,n}$ forming a path $P_2 := (O_{2,1}, \ldots, O_{2,n})$ where *all* objects $O_{2,\nu}$ are optimal for a_ν, $\nu = 1, \ldots, n$.
(Hence P_2 is "better" than P_1 because all nodes of P_2 are optimal while the start node of P_1 is possibly not optimal. — By the way, the case of $n = 2$ means the original OSP with $X_i := O_{i,1}$ and $Y_i := O_{i,2}$, $i = 1, 2$.)

Proof: We recursively construct the prefixes

$$P_{2,\nu} := (O_{2,1}, \ldots, O_{2,\nu})\,,\ \nu = 1, \ldots, n\,.$$

For $\nu = 1$, the path $P_{2,1} = (O_{2,1})$ is already given by the assumption that $O_{2,1}$ is optimal.
If $P_{2,\nu}$ exists then $P_{\nu+1}$ is constructed as follows: Let $X_i := O_{i,\nu}$, $i = 1, 2$ and let $Y_1 := O_{1,\nu+1}$; moreover, define $x := a_\nu$ and $y := a_{\nu+1}$. Then the objects X_1, X_2, Y_1, x, y satisfy the conditions of the definition of the OSP. Hence there exists an optimal successor $Y_2 =: O_{2,\nu+1}$ of $X_2 = O_{2,\nu}$, and the path $P_{2,\nu+1}$ is ready.
Finally, let $P_2 := P_{2,n}$. ∎

The next result says that the OSP implies the weak Bellman property. As a consequence, order preservance is at least as strong as the weak Bellman property. This fact is already mentioned in [9], p. 670 in the special case that $\mathcal{U}$ is a set of paths in a decision graph.

Theorem 2.4. Given an RDM $\Xi = (\mathcal{U}, \mathcal{A}, R, C, \mathcal{G}_\mathcal{U})$ such that $\mathcal{G}_\mathcal{U}$ is finite and acyclic and Ξ has optimal successor property.
Then Ξ has the weak Bellman property.

Proof: We start with the following definition: Given $a \in \mathcal{A}$ and an object $O \in R^{-1}[a]$, which is possibly not optimal. Then for all paths $P = (O_* = O_0, O_1, \ldots, O_m = O)$ in $\mathcal{G}_\mathcal{U}$ we define $k(P)$ as the length of the longest uninterrupted sequence of optimal objects at the end of P. More precisely, $k(P) = m + 1$ if every O_κ is optimal for some $a_\kappa \in \mathcal{A}$, $\kappa = 0, \ldots, m$; otherwise there exists a *maximal* $l \leq m$ such that O_l is not minimal for any $a' \in \mathcal{A}$; then $k(P) := m - l$.

In $\mathcal{G}_\mathcal{U}$ there exist only finitely many paths because $\mathcal{G}_\mathcal{U}$ is finite and acyclic. Hence we can find a path $P = P^*$ between O_* and some $O \in R^{-1}[a]$ for which $k(P)$ is maximal. We now prove that P^* only consists of optimal objects.
Otherwise P^* can be written as $Q_1 \oplus P_1$ where

- $P_1 = (O_{1,1}, \ldots, O_{1,n})$ and $n = k(P^*) + 1$,
- $O_{1,\nu} \in R^{-1}[a_\nu]$ $(\nu = 1, \ldots, n)$ for some $a_1, \ldots, a_n \in \mathcal{A}$ with $a_n = a$,
- $O_{1,2}, \ldots O_{1,n}$ are optimal for $a_2, \ldots, a_n$, respectively.

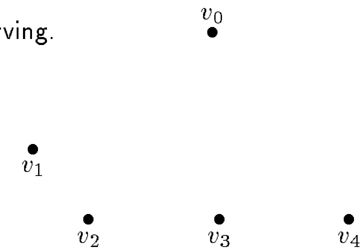
Remark 1.1.(ii) yields an object $O_{2,1} \in \mathcal{U}$ that is optimal for a_1. Then *Lemma 2.3* says that there exists a path $P_2 := (O_{2,1}, \ldots, O_{2,n})$ consisting of optimal objects $O_{2,i}$ for a_i, $i = 1, 2 \ldots, n$. In particular,

(1) P_2 has $n = k(P^*) + 1$ nodes, and $O_{2,n}$ is optimal for $a_n = a$.

Then Q_2 be a path from O_* to $O_{2,1}$ and let $P^{**} := Q_2 \oplus P_2$. Then (1) implies that $k(P^{**}) > n = k(P^*)$, which is a contradiction to the maximality of $k(P^*)$.

Hence P^* indeed consists of optimal nodes, and the weak Bellman property is proven. ∎

Remark 2.5. If a RDM Ξ has the matroid property, then Ξ is not always order preserving.



Proof: Recalling *Example 1.3*, we define the graph $G = (V, E)$ with

$$V := \{v_0, \ldots, v_4\}, \quad E := \{r_i = \{v_0, v_i\} \mid i = 1, \ldots, 4\}.$$

Let $\gamma(r_i) := 20 - i^2$ $(i = 1, \ldots, 4)$ and let $O_1 := \{r_1, r_4\}$, $O_2 := \{r_2, r_3\}$.
We define the RDM Ξ as in *Example 1.3*.
Then $C(O_1) = 23 < 27 = C(O_2)$. The best successor O_i' of O_i with $R(O_i') = |O_i'| = 3$ is constructed by adding the optimal missing edge to O_i $(i = 1, 2)$.
Consequently,

$$\mu(O_1, 3) \;=\; C(\{r_1, r_3, r_4\}) \;=\; 34 \;>\; 31 \;=\; C(\{r_2, r_3, r_4\}) \;=\; \mu(O_2, 3)$$

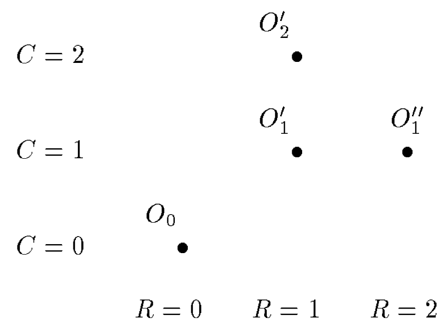
so that Ξ is indeed not order preserving. ∎

Remark 2.6. On the other hand, if Ξ is order preserving then Ξ does not always have the matroid property; this is even true if R is C-monotone.

Proof: We define the following RDM $\Xi = (\mathcal{U}, \mathcal{A}, R, C, \mathcal{G}_\mathcal{U})$: Let $\mathcal{U} := \{O_0, O_1', O_2', O_1''\}$ and $\mathcal{A} := \{0, 1, 2\}$ with $O_* := O_0$ and $a_* := 0$. Moreover, let

$$\mathcal{E}_\mathcal{U} := \{ (O_0, O_1'), (O_0, O_2'), (O_1', O_1'') \},$$

$$\begin{array}{llll} R(O_0) = 0, & R(O_1') = 1, & R(O_2') = 1, & R(O_1'') = 2, \\ C(O_0) := 0, & C(O_1') := 1, & C(O_2') := 2, & C(O_1'') := 1. \end{array}$$



Then it is easy to see that Ξ is order preserving and that R is C-monotone. On the other hand, for $V_1 := O_2'$ there exists no $Z \in \mathcal{U}$ with $e_1 := (V_1, Z) \in \mathcal{E}_\mathcal{U}$ in spite of the existence of the arc $e_0 := (V_0, W) := (O_1', O_1'')$. Therefore Ξ does not have the matroid property. ∎

Remark 2.7. Note that in the situations of *Remarks 2.5* and *2.6* the optimal successor property is given (see *Theorems 2.1, 2.2*). Therefore the OSP is not strong enough to yield order preservance and C-monotonity; moreover, the OSP does not always imply the matroid property. ∎

The next result describes the relationship of the strong Bellman condition to the optimal successor property.

Theorem 2.8.

a) The optimal successor property does not always imply the strong Bellman condition.

b) The strong Bellman condition does not always imply the optimal successor property.

c) If the strong Bellman condition is given and every set $R^{-1}[a]$, $a \in \mathcal{A}$ has a unique minimum, then the OSP is given.

Proof to a): We define the following RDM $\Xi_0 = (\mathcal{U}, \mathcal{A}, R, C_0, \mathcal{G}_\mathcal{U})$: Let $\mathcal{U} := \{O_0, O_1, O_1', O_2, O_2'\}$ and $\mathcal{A} := \{0, 1, 2\}$. R is defined as a function: Let $R(O_i) = i$ and $R(O_i') := i$ for all i. We define $C(O_1') := 1$ and $C(O) := 0$ for all $O \in \mathcal{U} \backslash \{O_1'\}$. The set $\mathcal{E}_\mathcal{U}$ of all arcs of $\mathcal{G}_\mathcal{U}$ is defined as

$$\mathcal{E}_\mathcal{U} \;:=\; \{ (O_0, O_1), (O_0, O_1'), (O_1, O_2), (O_1', O_2') \}.$$

Then Ξ_0 has the optimal successor property. We only must consider $\mathcal{N}_{\mathcal{U}}(O_1)$ and $\mathcal{N}_{\mathcal{U}}(O_1')$. Note that $(O_1', O_2') \in \mathcal{E}_{\mathcal{U}}$ where O_2' is optimal for $2 \in \mathcal{A}$; so we must check whether the optimal element O_1 has an optimal successor, too. But this is true because O_2 is an optimal successor of O_1.

The strong Bellman property, however, is not given; the path (O_0, O_1', O_2') ends at the optimal object O_2' but the intermediate object O_1' is not optimal.

Proof to b): We define the RDM $\Xi_1 = (\mathcal{U}, \mathcal{A}, R, C_1, \mathcal{G}_{\mathcal{U}})$ such that all components except the cost measure are the same as in *Part a)*. Let $C(O_2') := 1$ and $C(O) := 0$ for all $O \neq O_2'$.
Then the strong Bellman condition is satisfied; for this we consider all paths from O_0 to an optimal object:

$$(O_0)\,,\ \ (O_0, O_1)\,,\ \ (O_0, O_1, O_2)\ \text{ and }\ (O_0, O_1')\,.$$

It is easy to see that each of these paths exclusively consists of objects X that are optimal for $R(X) \in \mathcal{A}$. The optimal successor property, however, is not given, because O_1 has the optimal successor O_2, but the optimal object O_1' does not have any optimal successor.

Proof to c): Given two elements $x, y \in \mathcal{A}$ and the objects $X_1, X_2 \in R^{-1}[x]$, $Y_1 \in R^{-1}[y]$. Let $(X_1, Y_1) \in \mathcal{E}_{\mathcal{U}}$ and let X_2, Y_1 be optimal for x and y, respectively. We show that $X_1 = X_2$.
Condition *(iii)* of *Remark 1.1* guarantees a path P from O_* to X_1. Let $Q := P \oplus (X_1, Y_1)$. Then Q ends with the optimal object Y_1. So we can apply the strong Bellman property to the object $X_1 \in \mathcal{U}$ on the path Q. Hence X_1 is optimal for x, and the same was assumed for X_2; the uniqueness of optimal objects implies that $X_1 = X_2$.
Hence $(X_2, Y_1) = (X_1, Y_1) \in \mathcal{E}_{\mathcal{U}}$ so that X_2 has the optimal successor $Y_2 := Y_1$. ∎

Altogether, the assertions of the following diagram are true:

Ξ is order preserving,
R is C-monotone

Ξ has matroid property

Ξ has the strong Bellman property,
each $a \in \mathcal{A}$ has a unique minimal object

Ξ has the optimal successor property

Ξ has the weak Bellman property

3. The Dynamic Programming Algorithm DP_1

In this paragraph we present a dynamic programming algorithm for relational decision models. We show that DP_1 executes the same steps as FORD-BELLMAN and GREEDY if DP_1 is applied to the optimal path problem and the minimal spanning tree problem, respectively. After this we prove the correctness of DP_1; at last we consider its complexity in the general case and in the *Examples 1.1 — 1.4*.

When introducing DP_1 we assume that the conditions of *Remark 1.1* are given, and we make the following additional assumptions:

(I) $\mathcal{G}_{\mathcal{U}}$ is finite and acyclic; $\mathcal{A} = \{a^{(0)}, a^{(1)}, \ldots, a^{(n)}\}$ is also finite, and $a_* = a^{(0)}$.

(II) The set $\mathcal{A}$ is orderable with respect to $\mathcal{G}_{\mathcal{U}}$ and R; in particular, the given enumeration $a^{(i)}$ is admissible.

(III) Ξ has the optimal successor property.

(IV) The start object O_* is given.
Moreover, we assume that $R^{-1}[a_*] = R^{-1}\left[a^{(0)}\right] = \{O_*\}$. and that $\left(O^*, a^{(k)}\right) \notin R$ for all $k > 0$.
This situation can easily be generated by adding artificial start objects as described in *Remark 1.15.b)*.

Our algorithm has the same global structure as the usual dynamic programming methods:

1. Initialize $\psi\left(a^{(0)}\right) := O_*$.
2. FOR $k = 1$ TO n DO
Choose $\psi\left(a^{(k)}\right)$ as the optimal object within the set $\mathcal{M}$ of candidates; this set $\mathcal{M}$ is constructed by generating all $\mathcal{G}_{\mathcal{U}}$-successors of the optimal objects $\psi\left(a^{(j)}\right)$, $j < k$, which have earlier been found.

3.1. The Formulation of the Procedure DP$_1$

We next give the detailed description of algorithm DP$_1$. The numbers of the steps are the same as in FORD-BELLMAN. The look-ahead step 2.0 of DP$_1$ often helps to save time.

PROCEDURE DP$_1$

1. Initialize $\psi(a^{(0)}) := O_*$ and $\mathcal{L} := \mathcal{L}_0 := \left\{\psi\left(a^{(0)}\right)\right\}$.
Moreover, generate an empty list $\mathbf{L}(a^{(l)})$ for every element $l = 1, \ldots, n$
2. (Computation of $\psi\left(a^{(k)}\right)$, $k = 1, \ldots, n$; the objects $\psi\left(a^{(j)}\right)$, $j = 0, \ldots, k-1$ are already given in the k-th step; they can be found in the set $\mathcal{L} = \mathcal{L}_{k-1} = \left\{\psi\left(a^{(0)}\right), \ldots, \psi\left(a^{(k-1)}\right)\right\}$.)

FOR $k := 1$ TO n DO

2.0 For all l with $a^{(l)} \in R\left[\mathcal{N}_{\mathcal{U}}(\psi(a^{(k-1)}))\right]$ do
Compute an object $O^{(k-1,l)} \in \mathcal{U}$ with
$\left(\psi(a^{(k-1)}, O^{(k-1,l)}\right) \in \mathcal{E}_{\mathcal{U}}$, $\left(O^{(k-1,l)}, a^{(l)}\right) \in R$ and $C\left(O^{(k-1,l)}\right) = \mu\left(\psi(a^{(k-1)}), a^{(l)}\right)$.
Insert the pair $(k-1, O^{(k-1,l)})$ into the list $\mathbf{L}(a^{(l)})$.

2.1 Construct $X := \left\{j \in \{0, \ldots, k-1\} \,\middle|\, (\exists\, O \in \mathcal{U})\ (\psi(a^{(j)}), O) \in \mathcal{E}_{\mathcal{U}},\ (O, a^{(k)}) \in R\right\}$.

2.2 Construct $\mathcal{M} := \left\{O^{(j,k)} \,|\, j \in X\right\}$.

2.3 Choose $\psi(a^{(k)}) \in \mathcal{M}$ such that
$C\left(\psi(a^{(k)})\right) := \min\{C(O) \,|\, O \in \mathcal{M}\}$.

2.4 $\mathcal{L} := \mathcal{L} \cup \left\{\psi\left(a^{(k)}\right)\right\}$.

3.2. Elementary Properties DP$_1$

We first see that in *Example 1.1*, FORD-BELLMAN and DP$_1$ do the same things. This is an immediate consequence of the following observation:

Remark 3.1. Given the notations of *Example 1.1.* For all $j < k$, the following assertions are equivalent:

(i) $a^{(j)} \in N^-\left(a^{(k)}\right)$.

(ii) $\left(\exists\, O \in \mathcal{P}\left(a^{(0)}\right) = \mathcal{U}\right)\ \left(\psi(a^{(j)}), O\right) \in \mathcal{E}_{\mathcal{U}} \wedge \left(O, a^{(k)}\right) \in R$.

Proof: If (i) is true then $r := \left(a^{(j)}, a^{(k)}\right) \in E$, and $O := \psi(a^{(j)}) \oplus r$ verifies assertion (ii).

If assertion (ii) is given then $\left(\psi(a^{(j)}), O\right) \in \mathcal{E}_{\mathcal{U}}$ implies that there exists $r \in E$ with $O = \psi(a^{(j)}) \oplus r$; in particular $\alpha(r) = \omega\left(\psi(a^{(j)})\right) = a^{(j)}$. Moreover, (ii) implies that $\left(O, a^{(k)}\right) \in R$ so that $\omega(r) = \omega(O) = a^{(k)}$. Hence $r = \left(a^{(j)}, a^{(k)}\right)$, where $r \in E$, and Assertion (i) is proven. ■

We next compare DP_1 to GREEDY:

Remark 3.2. If DP_1 is applied to *Example 1.3* then it emulates Procedure GREEDY.

Proof: Let $k \in \{1, \ldots, n\}$. We first consider the set X constructed in Step 2.1 of DP_1. For all j and O, the condition $\left(\psi\left(a^{(j)}\right), O\right) \in \mathcal{E}_{\mathcal{U}}$ implies that O is generated by adding one element to $\psi\left(a^{(j)}\right)$ so that $|O| = \left|\psi\left(a^{(j)}\right)\right| + 1 = j+1$; if, in addition, $\left(O, a^{(k)}\right) \in R$ then $|O| = k$ so that $j = |O| - 1 = k - 1$. Consequently, Step 2.1 generates $X = \{k-1\}$. It follows that $\mathcal{M}$ only consists of a single forest $O^{(k-1,k)}$, which has the property $C\left(O^{(k-1,k)}\right) = \mu\left(\psi\left(k-1\right), k\right)$. This means that $O^{(k-1,k)}$ has been generated by adding a minimal feasible edge to $\psi\left(k-1\right)$. Consequently, Step 2.4 of DP_1 finds the same forest $\psi(k) = O^{(k-1,k)}$ as Step 2 of GREEDY. (This conclusion is based on the following assumption: If there are several feasible edges with equal costs then DP_1 and GREEDY chose the same edge.) ■

We next consider the correctness and the complexity of DP_1. For this purpose we make several simple observations:

Remark 3.3. The following definitions are equivalent to step 2.2:

(i) $X := \left\{ j \in \{0, \ldots, k-1\} \,\middle|\, a^{(k)} \in R\left[\mathcal{N}_{\mathcal{U}}\left(\psi\left(a^{(j)}\right)\right)\right] \right\}$.

(ii) $X := \left\{ j \in \{0, \ldots, k-1\} \,\middle|\, R^{-1}\left[a^{(k)}\right] \cap \mathcal{N}_{\mathcal{U}}\left(\psi\left(a^{(j)}\right)\right) \neq \emptyset \right\}$. ■

Lemma 3.4. Let $\hat{\jmath} < \hat{k}$. Then the following assertions are equivalent:

(a) $a^{(\hat{k})} \in R\left[\mathcal{N}_{\mathcal{U}}(\psi(a^{(\hat{\jmath})}))\right]$.

(b) $\hat{\jmath} \in X$ in the $\hat{k}$-th iteration.

(c) A pair $\left(\hat{\jmath}, O^{(\hat{\jmath},\hat{k})}\right)$ is inserted into $\mathbf{L}\left(a^{(\hat{k})}\right)$ in Step 2.0 of the $(\hat{\jmath}+1)$st round.

(d) $R^{-1}\left[a^{(\hat{k})}\right] \cap \mathcal{N}_{\mathcal{U}}\left(\psi(a^{(\hat{\jmath})})\right) \neq \emptyset$.

Proof: The equivalence of (a) and (b) follows from *Remark 3.3 (i)*. The assertion (a) $\Longleftrightarrow$ (c) is clear by the description of Step 2.0 for $\hat{\jmath} := k-1$ and $\hat{k} := l$. Finally, (a) is trivially equivalent to (d). ■

3.3. The Correctness of $\mathbf{DP}_1$

We next show that DP_1 outputs optimal objects $\psi(a^{(k)})$, $k = 0, \ldots, n$.

Theorem 3.5. *(Correctness of DP_1)*
Given a relational decision model $\Xi = (\mathcal{U}, \mathcal{A}, R, C, \mathcal{G}_{\mathcal{U}})$. We assume that Ξ satisfies the conditions (I) – (IV). Then the object $\psi\left(a^{(k)}\right)$ found by DP_1 is optimal for $a^{(k)}$, $k = 0, \ldots, n$.

Proof (Induction on k): If $k = 0$ then the object $\psi\left(a^{(k)}\right) = O_*$ is optimal for $a^{(k)} = a^{(0)}$ because $R^{-1}\left[a^{(0)}\right] = \{O_*\}$ by *Condition IV*.
Now let $1 \leq k \leq n$; we assume that our assertion is true for all $0 \leq j < k$. Then *Remark 1.1.(ii)* yields an object O that is optimal for $a^{(k)}$. Then $O \neq O_*$ as $O \in R^{-1}\left[a^{(k)}\right]$ while O_* is not in $R^{-1}\left[a^{(k)}\right]$ because of *Condition IV)*. *Remark 1.1.(iii)* yields a path P from O_* to O. The path P has at least one arc because $O \neq O_*$. Let $\left(\overline{O}, O\right)$ be the last arc of P and let $a^{(j)} \in R\left[\,\overline{O}\,\right]$. (Recall that R is total). Then $j \in X$ because of the admissibility of the ordering $\left(a^{(\nu)}\right)_{\nu=0}^{n}$; hence Step 2.0 finds an object $O^{(j,k)}$ (see *Lemma 3.4, (b)* $\Longleftrightarrow$ *(c)*).
Note now that $\overline{O} \in R^{-1}\left[a^{(j)}\right]$ has the successor O which is optimal for $a^{(k)}$; moreover, $\psi\left(a^{(j)}\right)$ is optimal for $a^{(j)}$ by the assumption of the induction. So the OSP yields an $\widetilde{O} \in \mathcal{N}_{\mathcal{U}}\left(\psi\left(a^{(j)}\right)\right)$ such that $\widetilde{O}$ is optimal for $a^{(k)}$. Consequently, $O^{(j,k)}$ with $C\left(O^{(j,k)}\right) = \mu\left(\psi(a^{(j)}), a^{(k)}\right) \leq C(\widetilde{O})$ is likewise optimal for $a^{(k)}$. Moreover, $O^{(j,k)} \in \mathcal{M}$ in the k-th iteration because $j \in X$ (see above); hence Step 2.3 indeed finds an optimal object $\psi\left(a^{(k)}\right)$ for $a^{(k)}$. ■

We next summarize several sufficient conditions for the correct behaviour of DP_1, and we discuss how they are satisfied in *Example 1.1 – 1.5*.

Corollary 3.6. Given a relational decision process $(\mathcal{U}, \mathcal{A}, R, C, \mathcal{G}_{\mathcal{U}})$ and an ordering $\vec{\mathcal{A}} := (a^{(0)}, \ldots, a^{(n)})$ of $\mathcal{A}$. The procedure DP_1 works correctly if the ordering $\vec{\mathcal{A}}$ is admissible and one of the following conditions (i) – (iv) is satisfied:

(i) Ξ is order preserving and R is C-monotone. (Recall *Theorem 2.1* and *3.5.*)

(ii) Ξ has the matroid property. (Recall *Theorem 2.2* and *3.5.*)

(iii) Ξ has the strong Bellman property and every $a \in \mathcal{A}$ has a unique optimal object. (Recall *Theorem 2.8.c)* and *3.5.*)

(iv) Ξ has the optimal successor property. (Recall *Theorem 3.5.*)

We next check the correctness of DP_1 in the *Examples 1.1 – 1.5.*

Example 1.1: We assume that the cost measure C is order preserving in graph theoretical sense (see *Remark 1.2*). As shown in *Remark 1.15.a)*, topological sorting of the set V of vertices yields an admissible ordering of $\mathcal{A} = V$. *Remark 1.4.a)* implies that RDM Ξ constructed in *Example 1.1* is order preserving, and *Remark 1.14.a)* says that R is C-monotone. So *Condition (i)* is given, and DP_1 works correctly. More precisely, for all $j = 0, \ldots, n$, the path $\psi(a^{(j)})$ output by DP_1 has minimal costs C among all paths from a_* to $a^{(j)} \in \mathcal{A} = V$.

Example 1.2: Let $\Xi = (\mathcal{U}, \mathcal{A}, R, C, \mathcal{G}_{\mathcal{U}})$ be the RDM constructed in *Example 1.2*.
We assume that there exists an admissible ordering of $\mathcal{A}$. Moreover, we assume that one of the following conditions is given:

- The cost measure C is constructed according to *Remark 1.4.b)* so that Ξ is order preserving.
  In addition, R is C-monotone.
  Then *Condition (i)* is satisfied.
- The cost measure C is defined according to *Remark 1.10.*
  Then Ξ has the optimal successor property, and *Condition (iv)* is satisfied.

In both situations, DP_1 works correctly. This means that for all $j = 0, \ldots, n$, the path $\psi(a^{(j)})$ output by DP_1 has minimum costs C among all paths starting at a_* and ending within the node set $a^{(j)} \subseteq V$.

Example 1.3: Let Ξ be the RDM constructed in *Example 1.3*. The definition $a^{(j)} := j,\ j = 0, \ldots, n$ yields an admissible ordering of $\mathcal{A}$ (see *Remark 1.15.a)*. Recalling *Remark 1.6* and *Definition 1.7* we see that the original matroid property for forests implies that the RDM Ξ has the matroid property. So *Condition (ii)* is satisfied, and DP_1 outputs an optimal forests $\psi(a^{(j)})$ for each given cardinality j.

Example 1.4: Let Ξ be the RDM defined in *Example 1.4*. It follows from *Remark 1.15.a)* that the definition $a^{(j)} := 1991 + j,\ j = 0, \ldots, 9$ yields an admissible ordering of $\mathcal{A}$. *Remark 1.4.d)* yields the order preservance of Ξ, and *Remark 1.14.a)* says that R is C-monotone. So *Condition (i)* is given, and DP_1 works correctly. More precisely, for all $j = 0, \ldots, n$, the algorithm DP_1 finds a pair $\psi(a^{(j)}) = (a^{(j)}, y^{(j)})$ such that $y^{(j)}$ is the maximum profit the saver can achieve in the year $a^{(j)}$.

Example 1.5: Let $\Xi = (\mathcal{U}, \mathcal{A}, R, C, \mathcal{G}_{\mathcal{U}})$ be the RDM defined in *Example 1.5*.
We assume that $a^{(0)}, \ldots, a^{(n)}$ is an admissible ordering of $\mathcal{A}$. Moreover, we assume that h satisfies condition (MA) in *Remark 1.4.e)*.

Then Ξ is order preserving. The relation R is automatically C-monotone as shown in *1.14.a)*. Hence *Condition (i)* is satisfied so that DP_1 works correctly. This means that for all $j = 0, \ldots, n$ the following is true: The sequence $\psi(a^{(j)})$ of decisions has minimal costs C among all sequences effecting the state $a^{(j)}$. ∎

3.4. The Complexity of DP_1

We now consider the time consumed by DP_1.

Lemma 3.7. Let $T_i^{(k)}$ be the time consumed in Step 2.i of the k-th iteration of DP$_1$ ($i = 0, \ldots, 4\,,\ k = 1, \ldots, n$). Then the time T consumed by the procedure DP$_1$ is in

$$\mathcal{O}\left(\sum_{k=1}^{n}\left(T_0^{(k)} + T_1^{(k)} + T_2^{(k)} + T_3^{(k)}\right)\right) .$$

The *proof* is straight-forward and based on the fact that Steps 1 and 2.4 of DP$_1$ are irrelevant for T. ∎

We next investigate the complexity of DP$_1$ in the *Examples 1.1. — 1.4.*

Remark 3.8. In *Example 1.1* and *1.3*, the procedure DP$_1$ works as fast as FORD-BELLMAN and KRUSKAL, respectively. In *Example 1.2* and *1.4*, the time consumed by DP$_1$ is often substantially smaller than the number of the arcs of G and $\mathcal{G}_\mathcal{U}$, respectively. This is seen in the following discussion of the *Examples 1.1 – 1.4.*

Example 1.1:

We make the following assumption:

(+) For each $a^{(k)} \in V$ the lists $\Lambda^-(a^{(k)})$ and $\Lambda(a^{(k)})$ are given; they contain all nodes of $N^-\left(a^{(k)}\right)$ and $N\left(a^{(k)}\right)$, respectively.

Then DP$_1$ comes out with $\mathcal{O}(|E|)$ steps; this is equal to the complexity of the usual Ford-Bellman Procedure for acyclic graphs.

Proof: We consider the quantities $T_0^{(k)}$ $(i = 0, 1, 2, 3)$ for all $k = 1, \ldots, n$:

Let us first calculate $T_0^{(k)}$. It is easy to see that every path $O \in \mathcal{U}$ and for all $l \geq k$ the following is true:

(1) $\Big(O \in \mathcal{N}_\mathcal{U}\left(\psi\left(a^{(k-1)}\right)\right) \ \text{and}\ \left(O, a^{(l)}\right) \in R\Big) \quad \Longleftrightarrow$

O is of the form $O = \psi\left(a^{(k-1)}\right) \oplus \left(a^{(k-1)}, a^{(l)}\right)$ where $a^{(l)} \in N\left(a^{(k-1)}\right)$.

This implies that $a^{(l)} \in R\left[\mathcal{N}_\mathcal{U}\left(\psi\left(a^{(k-1)}\right)\right)\right]$ iff $\left(a^{(k-1)}, a^{(l)}\right) \in E$ so that all of these l's can be found by traversing $\Lambda\left(a^{(k)}\right)$ in $\left|N\left(a^{(k-1)}\right)\right|$ time units. A further consequence of (1) is that $\psi\left(a^{(k-1)}\right) \oplus \left(a^{(k-1)}, a^{(l)}\right)$ forms already the desired path $O^{(k-1,l)}$ so that

$$T_0^{(k)} = \left|N\left(a^{(k-1)}\right)\right| .$$

We first calculate $T_1^{(k)}$. For this we note that $X = \left\{j \;\middle|\; a^{(j)} \in N^-\left(a^{(k)}\right)\right\}$. This and *Assumption* (+) imply that

$$(2) \qquad T_1^{(k)} = \left|N^-\left(a^{(k)}\right)\right| .$$

To estimate $T_2^{(k)}$ we note that Step 2.2 generates $\mathcal{M} = \left\{\psi\left(a^{(j)}\right) \oplus \left(a^{(j)}, a^{(k)}\right) \;\middle|\; j \in X\right\}$; so the computation of this set takes $|X| \stackrel{(1)}{=} \left|N^-\left(a^{(k)}\right)\right|$ time units. Moreover, $T_3^{(k)} = |\mathcal{M}| = |X| = \left|N^-\left(a^{(k)}\right)\right|$.

Hence the time of computation is

$$\mathcal{O}\left(\sum_{k=1}^{n}\left(T_0^{(k)} + T_1^{(k)} + T_2^{(k)} + T_3^{(k)}\right)\right) \in \mathcal{O}\left(\sum_{k=1}^{n}\left|N\left(a^{(k-1)}\right)\right| + 3 \cdot \sum_{k=1}^{n}\left|N^-\left(a^{(k)}\right)\right|\right) = \mathcal{O}(|E|) .$$

Example 1.2:

Let us first recall *Remark 1.4.b)* and *1.10*. They show how to construct cost measures C such that the resulting RDM $\Xi = (\mathcal{U}, \mathcal{A}, R, C, \mathcal{G}_\mathcal{U}$ has the OSP.

We assume that $v_*^{(i)}$ is the end node of the path $\psi\left(a^{(i)}\right)$ for all $i = 0, \ldots, n$.

In order to estimate the time of computation, we start with a general observation: Let $i_1 < i_2$. Then a path

O is in $\mathcal{N}_{\mathcal{U}}\left(\psi(a^{(i_1)})\right) \cap R^{-1}\left[a^{(i_2)}\right]$ iff there exists a $w \in N(v_*^{(i_1)})$ such that $O = \psi\left(a^{(i_1)}\right) \oplus \left(v_*^{(i_1)}, w\right)$ and $w = \omega(O) \in a^{(i_2)}$. So it is easy to see that

$$(3) \qquad (\forall\, i_1 < i_2) \quad \left|\mathcal{N}_{\mathcal{U}}\left(\psi(a^{(i_1)})\right) \cap R^{-1}\left[a^{(i_2)}\right]\right| \;=\; \left|N(v_*^{(i_1)}) \cap a^{(i_2)}\right|.$$

We next consider an arbitrary $k \in \{1, \ldots, n\}$ and the number of entries in the list $\mathbf{L}(a^{(k)})$ immediately after Step 2.0 of the k-th iteration. Let $t(j,k) \in \{0,1\}$ the number of pairs $(j, O^{(j,k)})$ in the list $\mathbf{L}\left(a^{(k)}\right)$ for $j = 0, \ldots, k-1$. Recalling (c) $\Longleftrightarrow$ (d) of *Lemma 3.4* we easily see that $t(j,k) = 0$ and $t(j,k) = 1$ iff $\mathcal{N}_{\mathcal{U}}\left(\psi(a^{(j)})\right) \cap R^{-1}\left[a^{(k)}\right]$ is empty and nonempty, respectively. In both cases the following inequality is true:

$$(\forall\, 0 \le j < k) \quad \left|\mathcal{N}_{\mathcal{U}}\left(\psi(a^{(j)})\right) \cap R^{-1}\left[a^{(k)}\right]\right| \;\le\; t(j,k) \;=\; \left|(\{j\} \times \mathcal{U}) \cap \mathbf{L}(a^{(k)})\right|.$$

This implies that

$$(4) \qquad \left|\mathbf{L}\left(a^{(k)}\right)\right| \;=\; \sum_{j<k} t(j,k) \;\le\; \sum_{j<k}\left|\mathcal{N}_{\mathcal{U}}\left(\psi(a^{(j)})\right) \cap R^{-1}\left[a^{(k)}\right]\right| \;\stackrel{(3)}{=}\; \sum_{j<k}\left|N\left(v_*^{(j)}\right) \cap a^{(k)}\right|.$$

These facts are useful for estimating the computation times $T_0^{(k)}, \ldots, T_3^{(k)}$:

Let us first consider $T_0^{(k)}$. We assume that for every $O \in \mathcal{N}_{\mathcal{U}}(\psi(a^{(k)}))$ the set $R[O]$ is available and that Step 2.0 is executed as follows:

For all $O \in \mathcal{N}_{\mathcal{U}}\left(\psi(a^{(k-1)})\right)$
  For all l with $a^{(l)} \in R[O]$ do
    IF $O^{(k-1,l)}$ does not yet exist THEN $O^{(k-1,l)} := O$;
    IF $O^{(k-1,l)}$ already exists and $C\left(O^{(k-1,l)}\right) > C(O)$ THEN $O^{(k,l)} := O$.

This procedure takes a time of

$$T_0^{(k)} \;\in\; \mathcal{O}\left(\sum_{O \in \mathcal{N}_{\mathcal{U}}\left(\psi(a^{(k-1)})\right)} \left|R[O]\right|\right).$$

To estimate this sum we observe that

$$\sum_{O \in \mathcal{N}_{\mathcal{U}}\left(\psi(a^{(k-1)})\right)} \left|R[O]\right| \;=\; \left|\left\{(O,l)\;\middle|\; O \in \mathcal{N}_{\mathcal{U}}(\psi(a^{(k-1)}),\ l > k-1,\ (O, a^{(l)}) \in R\right\}\right|$$
$$= \textstyle\sum_{l>k-1}\left|\mathcal{N}_{\mathcal{U}}\left(\psi(a^{(k-1)}\right) \cap R^{-1}[a^{(l)}]\right| \;\stackrel{(3)}{=}\; \sum_{l>k-1}\left|N\left(v_*^{(k-1)}\right) \cap a^{(l)}\right|.$$

This implies that

$$T_0^{(k)} \;\in\; \mathcal{O}\left(\sum_{l>k-1}\left|N\left(v_*^{(k-1)}\right) \cap a^{(l)}\right|\right).$$

To estimate $T_1^{(k)}$ we conclude from the equivalence (b) $\Longleftrightarrow$ (c) of *Lemma 3.4* that X can be computed by scanning $\mathbf{L}\left(a^{(k)})\right)$ so that

$$T_1^{(k)} \;=\; \left|\mathbf{L}\left(a^{(k)}\right)\right| \;\stackrel{(4)}{\le}\; \sum_{j<k}\left|N\left(v_*^{(j)}\right) \cap a^{(k)}\right|.$$

In order to compute $T_2^{(k)}$ we again consider (b) $\Longleftrightarrow$ (c); so we can see that Step 2.2 can be executed by scanning all pairs $\left(j, O^{(j,k)}\right) \in \mathbf{L}\left(a^{(k)}\right)$. (It is not bad if an object $O^{(j',k)} = O^{(j'',k)}$ with $j' \ne j''$ occurs more than once in the representation of $\mathcal{M}$.) Consequently,

$$T_2^{(k)} \;=\; \left|\mathbf{L}\left(a^{(k)}\right)\right| \;\stackrel{(3)}{\le}\; \sum_{j<k}\left|N\left(v_*^{(j)}\right) \cap a^{(k)}\right|.$$

In particular, the representation of $\mathcal{M}$ consists of $\left|\mathbf{L}\left(a^{(k)}\right)\right|$ entries so that

$$T_3^{(k)} = \left|\mathbf{L}\left(a^{(k)}\right)\right| \overset{(4)}{\leq} \sum_{j<k}\left|N\left(v_*^{(j)}\right)\cap a^{(k)}\right| .$$

Altogether, the time T of computation lies in an order of magnitude which is given by

$$\sum_{k=1}^{n}\sum_{j<k}\left|N\left(v_*^{(j)}\right)\cap a^{(k)}\right| + \sum_{k=1}^{n}\sum_{l>k-1}\left|N\left(v_*^{(k-1)}\right)\cap a^{(l)}\right| .$$

Each of the two double sums is equal to $\displaystyle\sum_{0\leq i_1<i_2\leq n}\left|N\left(v_*^{(i_1)}\right)\cap a^{(i_2)}\right|$ so that

$$(5)\quad T \in \mathcal{O}\left(\sum_{0\leq i_1<i_2\leq n}\left|N\left(v_*^{(i_1)}\right)\cap a^{(i_2)}\right|\right) .$$

Before discussing this result more profoundly, let us have a look at the *data structure* that should be used to store the pairs $(i_1, O^{(i_1,i_2)})$. The estimation of $T_0^{(k)}$ is based on the assumption that inserting, searching and updating of $(i_1, O^{(i_1,i_2)})$ is possible in $\mathcal{O}(1)$ steps where $i_1 = k-1$. When estimating $T_1^{(k)}$ and $T_2^{(k)}$, we assume that the set $\mathbf{L}\left(a^{(i_2)}\right)$ with $i_2 = k$ can be scanned in $\left|\mathbf{L}\left(a^{(i_2)}\right)\right|$ steps.
This behavior can be effected by an array $\mathbf{Y}(i_1,i_2)$ where each element $y(i_1,i_2)$ is a record of three components: $y(i_1,i_2).1$ is equal to nil or contains the current pair $\left(i_1, O^{(i_1,i_2)}\right)$. If $y(i_1,i_2).1 \neq \mathrm{nil}$ then $y(i_1,i_2).2 = i_1^-$ and $y(i_1,i_2).3 = i_1^+$ where i_1^- and i_1^+ is the next index $i_1' < i_1$ and $i_1' > i_1$ for that $y(i_1',i_2).1 \neq \mathrm{nil}$, respectively.

Let us next compare the time T of DP_1 with the $\mathcal{O}(|E|)$ complexity of the usual Ford-Bellman strategy. For this we consider the special case that the sets $a^{(k)}$ are pairwise disjoint; this means that the relation R is a function $R : V \to \mathcal{A}$. Of course, if i_1 is given then the sets $N\left(v_*^{(i_1)}\right)\cap a^{(i_2)}$ $(i_2 = 1,\ldots,n)$ are also pairwise disjoint. Consequently, the following fact is true for all $i_1 = 0,\ldots,n-1$:

$$\sum_{i_2=i_1+1}^{n}\left|N\left(v_*^{(i_1)}\right)\cap a^{(i_2)}\right| = \left|\bigcup_{i_2<i_1+1}^{n}\left(N\left(v_*^{(i_1)}\right)\cap a^{(i_2)}\right)\right| = \left|N\left(v_*^{(i_1)}\right)\cap\left(\bigcup_{i_2=i_1+1}^{n} a^{(i_2)}\right)\right| = \left|N\left(v_*^{(i_1)}\right)\right| .$$

From this we can conlude that

$$\sum_{0\leq i_1<i_2\leq n}\left|N\left(v_*^{(i_1)}\right)\cap a^{(i_2)}\right| = \sum_{i_1=0}^{n-1}\sum_{i_2=i_1+1}^{n}\left|N\left(v_*^{(i_1)}\right)\cap a^{(i_2)}\right| = \sum_{i_1=0}^{n-1}\left|N\left(v_*^{(i_1)}\right)\right| .$$

This and (5) imply that

$$(6)\quad T \in \mathcal{O}\left(\sum_{i_1=0}^{n-1}\left|N\left(v_*^{(i_1)}\right)\right|\right)$$

Note that $V' := \left\{v_*^{(0)},\ldots,v_*^{(n-1)}\right\}$ is very often a proper subset of V so that $\sum_{i_1=0}^{n-1}\left|N\left(v_*^{(i_1)}\right)\right| = \sum_{v\in V'}|N(v)| < \sum_{v\in V}|N(v)| = |E|$. This and (6) make us expect that in many cases T is substantially smaller than $|E|$.

For example, assume that $a^{(0)} = \{s\}$ and all other sets $a^{(j)}$ have a cardinality within the interval $[0.5c, 2c]$ where $c = \sqrt{n}$. Then

$$(7)\qquad |V| = \sum_{j=0}^{n}\left|a^{(j)}\right| \in \left[1+\frac{1}{2}n\sqrt{n}\,,\,1+2\,n\sqrt{n}\right] .$$

We define

$$E := \{(v,w)\,|\,v\in a^{(i)}\,,\,w\in a^{(j)}\,,\,0\leq i<j\leq n\} .$$

Then

$$(8) \qquad |E| \;=\; \sum_{0\le i<j\le n} \left|a^{(i)}\right|\cdot\left|a^{(j)}\right| \;\ge\; \sum_{1\le i<j\le n} \left|a^{(i)}\right|\cdot\left|a^{(j)}\right| \;\ge\; \sum_{1\le i<j\le n} 0.5c\cdot 0.5c \;=\; \frac{n^2-n}{2}\cdot\frac{c^2}{4}\in\Theta(n^3).$$

On the other hand, $N\left(v_*^{(i_1)}\right) = a^{(i_1+1)}\cup\cdots\cup a^{(n)}$ for all i_1 so that $\left|N\left(v_*^{(i_1)}\right)\right| \le 2\cdot(n-i_1)\cdot\sqrt{n}$. Consequently,

$$(9) \qquad \sum_{i_1=0}^{n-1}\left|N\left(v_*^{(i_1)}\right)\right| \;\le\; \sum_{i_1=0}^{n-1} 2\cdot(n-i_1)\cdot\sqrt{n} \;=\; \frac{n^2+n}{2}\cdot\sqrt{n} \;\in\; \mathcal{O}(n^2\cdot\sqrt{n})\,.$$

Let us now compare (8) and (9). We see that the time $\sum_{i_1=0}^{n-1}\left|N\left(v_*^{(i_1)}\right)\right|$ consumed by DP_1 is $\sqrt{n} \overset{(7)}{\in} \Theta\left(\sqrt[3]{|V|}\right)$ times smaller than the complexity $|E|$ arising from first applying the Ford-Bellman Strategy to the graph G and then searching the paths $\psi(a^{(j)})$ $(j=0,\ldots,n)$ among all optimal paths from $v^{(0)}$ to v where $v\in a^{(j)}$. The reason is that Ford's and Bellman's algorithm considers *each* predecessor v of a given node w while DP_1 investigates v only if v belongs to the set V' of endpoints of optimal paths $\psi\left(a^{(k)}\right)$; in particular, for any set $a^{(k)}$ only one element $v=v_*^{(k)}$ is considered (and not all elements $v\in a^{(k)}$ as in the Ford-Bellman Procedure).

Example 1.3:
Let $a^{(k)} := k\,,\ k=0,\ldots,|V|-1 := n$.
We first consider Step 2.0 of the k-th iteration $(k\in\{0,\ldots,k-1\})$. The only l with $a^{(l)}\in R\left[\mathcal{N}_{\mathcal{U}}\left(\psi(a^{(k-1)})\right)\right] = R[\mathcal{N}_{\mathcal{U}}(\psi(k-1)]$ is $l=k$ because every forest O with $(\psi(k-1),O)\in\mathcal{E}_{\mathcal{U}}$ has k edges. This means that Step 2.0 only generates one object $O^{(k-1,l)}$, namely $O^{(k-1,k)}$.
In order to describe this step in more detail we assume that the arcs of E are already sorted by their γ-value, i.e.

$$E=\{e_1,\ldots,e_r\} \quad\text{and}\quad \gamma(e_1)>\ldots>\gamma(e_r)\,.$$

Moreover, we imagine that DP_1 calculates the index ρ of the current edge e_ρ; this index is initialized as $\rho:=1$ in Step 1. Every node $v\in V$ is associated with a value $cc(v)$ describing the connected component of v in the forest $\psi(k)$; $cc(v)=nil$ if v does not yet belong to $\psi(k)$. Then we can formulate Step 2.0 in the k-th iteration as follows:

2.0.1. WHILE (*Both end points of* e_ρ *belong to the same connected component of* $\psi(k-1)$) DO $\rho:=\rho+1$;

2.0.2 IF $\rho\le r$ THEN construct $\psi(k):=\psi(k-1)\cup\{e_\rho\}$;
FOR ALL $v\in V$ update $cc(v)$. (This is necessary if e_ρ connects two components of $\psi(k-1)$.)

Then *all* iterations of DP_1 cause altogether r executions of Step 2.0.1. Each Step 2.0.2 comes out with $\mathcal{O}(n)$ time and is executed $\le n$ times. Hence

$$(10) \qquad \sum_{k=1}^{n} T_0^{(k)} \;\in\; \mathcal{O}(r+n^2) \;=\; \mathcal{O}(n^2)\,.$$

Let us next consider $T_1^{(k)}$, $T_2^{(k)}$ and $T_3^{(k)}$ for $k=1,\ldots,n$: First $T_1^{(k)}=1$ because $X=\{k-1\}$. (Note that every forest $\psi\left(a^{(j)}\right)$, $j<k-1$ has less than $k-1$ elements and cannot have a forest of cardinality k as its successor.) Moreover, $T_2^{(k)}=1$ as $\mathcal{M}=\left\{O^{(k-1,k)}\right\}$, and $T_3^{(k)}=1$ for the same reason.
This means that the time complexity T of DP_1 is mainly influenced by $T_0^{(k)}$, i.e.

$$T \overset{(10)}{\in} \mathcal{O}\left(n^2\right)$$

This means that in this case DP_1 is as fast as the Greedy Procedure described in [7].

Example 1.4:
To calculate $T_0^{(k)},\ldots,T_3^{(k)}$ we make several observations. It is obvious that

(11)
a) The sets $\mathcal{B}^+\left(a^{(i_1)}\right)$, $i_1=0,\ldots,n-1$ are pairwise disjoint;
b) the sets $\mathcal{B}^-\left(a^{(i_2)}\right)$, $i_2=1,\ldots,n$ are pairwise disjoint;
c) the sets $\mathcal{B}^+\left(a^{(i_1)}\right)\cap\mathcal{B}^-\left(a^{(i_2)}\right)$, $i_1=0,\ldots,i_2-1$ are pairwise disjoint for every i_2.

We next note that for all i_1,

$$(12) \qquad \mathcal{N}_{\mathcal{U}}\left(\psi\left(a^{(i_1)}\right)\right) \;=\; \left\{\psi\left(a^{(i_1)}\right)\otimes b \,\middle|\, b\in\mathcal{B}^+\left(a^{(i_1)}\right)\right\}.$$

Moreover, if $b=\left(a^{(i_1)},a^{(i_2)},\eta,\delta\right)\in\mathcal{B}^+\left(a^{(k-1)}\right)$ then

$$(13) \qquad R\left(\psi\left(a^{(i_1)}\right)\otimes b\right) \;=\; a^{(i_2)}.$$

This implies that

$$(14) \qquad \mathcal{N}_{\mathcal{U}}\left(\psi\left(a^{(i_1)}\right)\right)\,\cap\,R^{-1}\left[a^{(i_2)}\right] \;=\; \left\{\psi\left(a^{(i_1)}\right)\otimes b \,\middle|\, b\in\mathcal{B}^+\left(a^{(i_1)}\right)\cap\mathcal{B}^-\left(a^{(i_2)}\right)\right\}.$$

We next estimate $T_0^{(k)}$. Applying (12) to $i_1 := k-1$ we see that Step 2.0 can be executed by traversing the set $\mathcal{B}^+\left(a^{(k-1)}\right)$. Moreover, let $b=\left(a^{(k-1)},a^{(l)},\eta,\delta\right)\in\mathcal{B}^+\left(a^{(k-1)}\right)$. Then Fact (13) with $i_2 := l$ says that $R\left(\psi\left(a^{(k-1)}\right)\otimes b\right) = a^{(l)}$ so that $\mathbf{L}\left(a^{(l)}\right)$ is updated.
Hence every bond $b\in\mathcal{B}^+\left(a^{(k-1)}\right)$ causes $O(1)$ steps so that

$$T_0^{(k)} \;\in\; \mathcal{O}\left(\left|\mathcal{B}^+\left(a^{(k-1)}\right)\right|\right).$$

We next consider $T_1^{(k)},\ldots,T_3^{(k)}$. For all j we define $t(j,k)\in\{0,1\}$ as the number of pairs $\left(j,O^{(j,k)}\right)$ occurring in $\mathbf{L}(a^{(k)})$. Moreover, *Fact (c)* $\iff$ *(d)* of *Lemma 3.4* says that $\left(j,O^{(j,k)}\right)\in\mathbf{L}\left(a^{(k)}\right)$ iff $\mathcal{N}_{\mathcal{U}}\left(\psi\left(a^{(j)}\right)\right)\cap R^{-1}\left[a^{(k)}\right]\neq\emptyset$, and Fact (14) implies that this is true iff $\mathcal{B}^+\left(a^{(j)}\right)\cap\mathcal{B}^-\left(a^{(k)}\right)\neq\emptyset$. Consequently,

$$(15) \qquad \left|\mathbf{L}\left(a^{(k)}\right)\right| \;=\; \sum_{j=1}^{k-1} t(j,k) \;\le\; \sum_{j=1}^{k}\left|\mathcal{B}^+\left(a^{(j)}\right)\cap\mathcal{B}^-\left(a^{(k)}\right)\right|.$$

Recalling the last assertion of (11) we obtain:

$$(16) \qquad \left|\mathbf{L}\left(a^{(k)}\right)\right| \overset{(15)}{\le} \sum_{j=1}^{k}\left|\mathcal{B}^+\left(a^{(j)}\right)\cap\mathcal{B}^-\left(a^{(k)}\right)\right| \;=\; \left|\bigcup_{j=0}^{k-1}\left(\mathcal{B}^+\left(a^{(j)}\right)\cap\mathcal{B}^-\left(a^{(k)}\right)\right)\right| \;=\; \left|\mathcal{B}^-\left(a^{(k)}\right)\right|.$$

Then the set X can be constructed by scanning $\mathbf{L}\left(a^{(k)}\right)$ so that

$$T_1^{(k)} \;=\; \left|\mathbf{L}\left(a^{(k)}\right)\right| \overset{(16)}{\le} \left|\mathcal{B}^-\left(a^{(k)}\right)\right|.$$

The set $\mathcal{M}$ is generated by traversing X so that

$$T_2^{(k)} \;\in\; \mathcal{O}(|X|) \;\subseteq\; \mathcal{O}\left(\left|\mathbf{L}\left(a^{(k)}\right)\right|\right) \overset{(16)}{\subseteq} \mathcal{O}\left(\left|\mathcal{B}^-\left(a^{(k)}\right)\right|\right).$$

At last,

$$T_3^{(k)} \;=\; |\mathcal{M}| \;\le\; |X| \;\le\; \left|\mathbf{L}\left(a^{(k)}\right)\right| \overset{(16)}{\le} \left|\mathcal{B}^-\left(a^{(k)}\right)\right|.$$

We next compute the time T consumed by DP_1. At first, $\sum_{k=1}^{n} T_0^{(k)}$ is in $\mathcal{O}\left(\sum_{k=1}^{n}\left|\mathcal{B}^+\left(a^{(k-1)}\right)\right|\right)$; moreover, Fact (11).a) implies that $\sum_{k=1}^{n}\left|\mathcal{B}^+\left(a^{(k-1)}\right)\right| = \left|\bigcup_{k=0}^{n-1}\mathcal{B}^+\left(a^{(k-1)}\right)\right| = |\mathcal{B}|$. Consequently

$$\sum_{k=1}^{n} T_0^{(k)} \;\in\; \mathcal{O}(|\mathcal{B}|).$$

For $i = 1,2,3$ our previous results imply that $\sum_{k=1}^{n} T_i^{(k)}$ is in $\mathcal{O}\left(\sum_{k=1}^{n}\left|\mathcal{B}^-\left(a^{(k)}\right)\right|\right)$; Fact (11).b) yields $\sum_{k=1}^{n}\left|\mathcal{B}^-\left(a^{(k)}\right)\right| = \left|\bigcup_{k=1}^{n}\mathcal{B}^-\left(a^{(k-1)}\right)\right| = |\mathcal{B}|$. Consequently

$$\sum_{k=1}^{n} T_i^{(k)} \;\in\; \mathcal{O}(|\mathcal{B}|).$$

Applying *Lemma 3.7* we obtain

$$T \;\in\; \mathcal{O}(|\mathcal{B}|)\,.$$

.

To get a better impression of the complexity $O(|\mathcal{B}|)$ we compare the quantity $|\mathcal{B}|$ with the number of all arcs in $\mathcal{G}_{\mathcal{U}}$. For this we make the following assumptions, which are not very restrictive:

(+) Let $O \in \mathcal{U}$. Whenever two different bonds $b_1 \neq b_2$ are sold in situation O, then the resulting financial situations $O \otimes b_1$ and $O \otimes b_2$ are different.

(++) There exists a year a with $\left|R^{-1}[a]\right| > 1$.

As seen in *Remark 1.4.d)*, for all $a \in \mathcal{A}$ and $O \in \mathcal{U}$ with $R(O) = a$ the following is true: $\mathcal{N}_{\mathcal{U}}(O) \;=\; \{O \otimes b \,|\, b \in \mathcal{B}^+(a)\}$. Consequently,

$$(17) \qquad (\forall\, a \in \mathcal{A})\,(O \in R^{-1}[a]\,) \quad |\mathcal{N}_{\mathcal{U}}(O)| \;=\; |\,\{O \otimes b \,|\, b \in \mathcal{B}^+(a)\}\,| \;\overset{(+)}{=}\; |\mathcal{B}^+(a)|\,.$$

This implies that

$$\begin{aligned} |\mathcal{E}_{\mathcal{U}}| \;&=\; \textstyle\sum_{O \in \mathcal{U}} |\mathcal{N}_{\mathcal{U}}(O)| \;=\; \sum_{a=1990}^{1999} \sum_{O \in R^{-1}[a]} |\mathcal{N}_{\mathcal{U}}(O)| \\ &\overset{(19)}{=}\; \textstyle\sum_{a=1990}^{1999} \sum_{O \in R^{-1}[a]} |\mathcal{B}^+(a)| \;=\; \sum_{a=1990}^{1999} |R^{-1}[a]| \cdot |\mathcal{B}^+(a)| \;\overset{(++)}{>}\; \sum_{a=1990}^{1999} |\mathcal{B}^+(a)| \;\overset{(11).a)}{=}\; |\mathcal{B}|\,. \end{aligned}$$

.

4. The Dynamic Programming Algorithm DP$_2$

We now present and investigate the dynamic programming procedure DP$_2$ for relational decision models $\Xi = (\mathcal{U}, \mathcal{A}, R, C, \mathcal{G}_{\mathcal{U}})$. In contrast to DP$_1$, the algorithm DP$_2$ works correctly even if the underlying relational decision model Ξ only satisfies the weak Bellman condition; by *Theorem 2.4* and *Remark 2.8 (b)*, this property is not so strong as the optimal successor property required in *Section 3*. Hence DP$_2$ is more universal than DP$_1$.

Given the RDM $\Xi = (\mathcal{U}, \mathcal{A}, R, C, \mathcal{G}_{\mathcal{U}})$. Then we define the function $\overline{\Psi}$ from $\mathcal{A}$ into the power set $2^{\mathcal{U}}$ of $\mathcal{U}$ as follows: For all a, $\Psi(a) \subseteq \mathcal{U}$ is the set of all objects that are optimal for a. More formally,

$$(\forall\, a \in \mathcal{A}) \quad \overline{\Psi}(a) \;:=\; \left\{ O \in \mathcal{U} \;\middle|\; (O, a) \in R \text{ and } C(O) = \min_{(O', a) \in R} C(O') \right\}\,.$$

We make the following assumptions:

(I) $\mathcal{G}_{\mathcal{U}}$ is finite and acyclic.

(II) The set $\mathcal{A}$ is orderable, and the given enumeration $\left(a^{(i)}\right)_{i=0}^{n}$ is an admissible ordering with respect to $\mathcal{G}_{\mathcal{U}}$ and R.

(III) Ξ has the weak Bellman's property.

(IV) The start object O_* is given.
Moreover, we assume that $R^{-1}[a_*] = R^{-1}\left[a^{(0)}\right] = \{O_*\}$. and that $\left(O^*, a^{(k)}\right) \notin R$ for all $k > 0$.
This situation can easily be generated by adding artificial start objects as described in *Remark 1.15.b)*.

Our task is to find an element $\psi(a) \in \overline{\Psi}(a)$ for all $a \in \mathcal{A}$. For this we consider the dynamic programming method DP$_2$. Its structure is similar to that of DP$_1$. The main difference is that the single object $\psi\left(a^{(k-1)}\right)$ and $\psi\left(a^{(k)}\right)$ is replaced by an object $\overline{O} \in \Psi\left(a^{(k-1)}\right)$ and $\overline{O} \in \Psi\left(a^{(k)}\right)$, respectively.

4.1. The Formulation of DP$_2$

We now give the detailed description of the procedure DP$_2$.

PROCEDURE DP$_2$

1. Define $\Psi\left(a^{(0)}\right) \;:=\; \overline{\Psi}(a^{(0)}) \;=\; \{O_*\}$ and
$$\mathcal{L} := \mathcal{L}_0 := \Psi(a^{(0)}) \;=\; \{O_*\}\,.$$

2. $\left(\begin{array}{l}\text{Computation of } \Psi(a^{(k)})\,, \quad k = 1,\ldots,n;\ \text{the sets } \Psi(a^{(j)})\,,\ j = 0,\ldots,k-1 \text{ are already given, and} \\ \mathcal{L}_k := \mathcal{L} = \bigcup_{j=0}^{k-1}\Psi(a^{(j)})\,.\end{array}\right)$

   FOR $k := 1$ TO n DO

   2.0 For all l with $a^{(l)} \in R\left[\mathcal{N}_{\mathcal{U}}\left(\Psi\left(a^{(k-1)}\right)\right)\right]$ do

   Compute the set $\mathbf{O}^{(k-1,l)}$ of all objects $\widetilde{O}$ with the following property:
$$\left(\exists\,\overline{O} \in \Psi\left(a^{(k-1)}\right)\right)\ \left((\overline{O},\widetilde{O}) \in \mathcal{E}_{\mathcal{U}},\ \left(\widetilde{O},a^{(l)}\right) \in R,\ C(\widetilde{O}) = \mu\left(\overline{O},a^{(l)}\right)\right)\,.$$

   Put each pair $(k-1,\widetilde{O})$ with $\widetilde{O} \in \mathbf{O}^{(k-1,l)}$ in the list $\mathbf{L}\left(a^{(l)}\right)$.

   2.1 Construct $X := \left\{j \in \{0,\ldots,k-1\}\ \middle|\ \left(\exists\,O \in \mathcal{U}\right)\left(\exists\,\overline{O} \in \Psi\left(a^{(j)}\right)\right)\ \left((\overline{O},O) \in \mathcal{E}_{\mathcal{U}} \wedge (O,a^{(k)}) \in R\right)\right\}$.

   2.2 Construct $\mathcal{M} := \bigcup\limits_{j\in X}\mathbf{O}^{(j,k)}$.

   2.3 Construct the following set $\Psi(a^{(k)}) \subseteq \mathcal{M}$:
$$\Psi(a^{(k)}) \;:=\; \left\{O \in \mathcal{M}\ \middle|\ C(O) = \min\{C(O')\,|\,O' \in \mathcal{M}\}\right\}.$$

   2.4 $\mathcal{L} := \mathcal{L} \cup \Psi\left(a^{(k)}\right)$.

3. FOR $k := 0$ TO n DO

   Choose an arbitrary object $\psi(a^{(k)}) \in \Psi(a^{(k)})$.

4.2. Elementary Properties of DP$_2$

The next remark describes an elementary observation about DP$_2$.

Remark 4.1. The following definitions are equivalent to step 2.2:

(i) $X \;:=\; \left\{j \in \{0,\ldots,k-1\}\ \middle|\ a^{(k)} \in R\left[\mathcal{N}_{\mathcal{U}}[\Psi(a^{(j)})]\right]\ \right\}$.

(ii) $X \;:=\; \left\{j \in \{0,\ldots,k-1\}\ \middle|\ R^{-1}\left[a^{(k)}\right] \cap \mathcal{N}_{\mathcal{U}}\left[\Psi\left(a^{(j)}\right)\right] \neq \emptyset\right\}$.

(iii) $X \;:=\; \left\{j \in \{0,\ldots,k-1\}\ \middle|\ \mathbf{O}^{(j,k)} \neq \emptyset\right\}$. ■

4.3. The Correctness of DP$_2$

We next show that the objects $\psi\left(a^{(j)}\right)$ found by DP$_2$ are actually optimal.

Theorem 4.2. Given an RDM $\Xi = (\mathcal{U},\mathcal{A},R,C,\mathcal{G}_{\mathcal{U}})$ satisfying the conditions (I) – (IV) at the beginning of this paragraph.
Then the objects $\psi(a^{(k)})$ found by DP$_2$ are indeed in $\overline{\Psi}(a^{(k)})$ for all $k = 0,\ldots,n$.

Proof: It is sufficient to show that

(1) $\Psi(a^{(k)}) \subseteq \overline{\Psi}(a^{(k)})\,,\ \ k = 0,\ldots,n.$

(2) $\Psi(a^{(k)}) \neq \emptyset\,, \;\; k = 1, \ldots, n.$

For this we first show an auxiliary result:

(3) For all $k = 0, \ldots, n$ the following is true:

$$(\forall\, O \in \mathcal{U}) \qquad O \in \Psi(a^{(k)}) \iff \text{There exists a Bellman path from } (O_*, a^{(0)}) \text{ to } (O, a^{(k)}).$$

The proof is given by an induction on k. The case of $k = 0$ is very easy. Note that $\overline{\Psi}(a^{(0)}) = \Psi(a^{(0)}) = \{O_*\}$. If $O \in \Psi\left(a^{(0)}\right)$ then $O = O_*$, and the path (O_*) only consisting of O_* is the desired Bellman path. On the other hand, given a Bellman path from $(O_*, a^{(0)})$ to $(O, a^{(k)})$ with $k = 0$. The $(O, a^{(0)}) \in R$ implies that $O = O_* \in \Psi\left(a^{(0)}\right)$.

We now assume that (3) is true for all $0 \leq j < k$ and prove this fact for k.

<u>*Proof to* " $\Longrightarrow$ "</u> : Let $O \in \Psi(a^{(k)})$. Recalling the construction of this set we obtain

(4) $(O, a^{(k)}) \in R.$

We now prove that

(5) $O \in \overline{\Psi}(a^{(k)})$, i.e., O is optimal for $a^{(k)}$.

For this purpose we compare O to an optimal element $\widehat{O} \in \overline{\Psi}\left(a^{(k)}\right)$ that can be reached by a Bellman path. More precisely, the weak Bellman property yields

- a path $\widehat{P} := \left(\widehat{O}_0, \ldots, \widehat{O}_s =: \widehat{O}\right)$ and
- elements $\widehat{a}_0 = a^{(0)}, \widehat{a}_1, \ldots, \widehat{a}_s = a^{(k)}$

such that $\widehat{O}_\sigma \in \overline{\Psi}(\widehat{a}_\sigma), \;\; \sigma = 0, \ldots, s.$

Then $O_* \neq \widehat{O}$; this follows from the relationships $\left(\widehat{O}, a^{(k)}\right) \in R$ and $(O_*, a^{(k)}) \notin R$ (see *Condition IV*)). The fact $O_0 = O_* \neq \widehat{O} = O_s$ implies $s > 0$. Since $s > 0$, the subscript $s - 1$ really occurs. Let j' such that $\widehat{a}_{s-1} = a^{(j')}$. We make the following observations:

(6) $\left(\widehat{O}_{s-1}, \widehat{a}_{s-1}\right) = \left(\widehat{O}_{s-1}, a^{(j')}\right) \in R.$

(7) $\left(\widehat{O}_s, a^{(k)}\right) = \left(\widehat{O}, a^{(k)}\right) \in R.$

(8) $\left(\widehat{O}_{s-1}, \widehat{O}_s\right) \in \mathcal{E}_{\mathcal{U}}\,.$

So the strict monotonity of R and Facts (6),(7),(8) imply that $j' < k$. Therefore we may apply direction " $\Longleftarrow$ " of (3) to j' and obtain:

(9) $\widehat{O}_{s-1} \in \Psi\left(a^{(j')}\right)$, i.e., $\widehat{O}_{s-1}$ is found by DP$_2$.

Moreover, consider the definition of X and let $O := \widehat{O}_s$ and $\overline{O} := \widehat{O}_{s-1}$; then (9), (8) and (7) imply that

(10) $j' \in X$ in Step 2.1 of the kth iteration.

These facts are now used to prove (5): It follows from Assertion (9) that Step 2.0 of the $(j'+1)$st iteration also considers $\overline{O} := \widehat{O}_{s-1}$; Facts (7) and (8) imply that $\mathcal{N}_{\mathcal{U}}(\overline{O}) \cap R^{-1}\left[a^{(i)}\right] \neq \emptyset$; consequently, Step 2.0 inserts at least one object $\widetilde{O}$ into $\mathbf{O}^{(j',k)}$ with the property $C(\widetilde{O}) = \mu\left(\widehat{O}_{s-1}, a^{(k)}\right)$. A further consequence of Facts (7) and (8) is that even $C(\widetilde{O}) = \mu\left(\widehat{O}_{s-1}, a^{(k)}\right) \leq C\left(\widehat{O}_s\right)$ so that $C(\widetilde{O}) = C\left(\widehat{O}_s\right)$ because $\widehat{O}$ is optimal for $a^{(k)}$. Moreover, $\widetilde{O} \in \mathbf{O}^{(j',k)} \stackrel{(10)}{\subseteq} \mathcal{M}$ so that the object $O \in \Psi\left(a^{(k)}\right)$ has been compared with $\widetilde{O}$ in Step 2.3 of the kth iteration.

Consequently, $C(O) \le C(\widetilde{O}) = C(\widehat{O}_s)$; this and the optimality of $\widehat{O}_s$ imply the same property of O.

We now have proven assertion (5) and must still construct a Bellman path from $(O_*, a^{(0)})$ to $(O, a^{(k)})$. For this we observe that $O \in \Psi\left(a^{(k)}\right)$ is only possible if $O \in \mathcal{M} = \bigcup_{j=0}^{k-1} \mathbf{O}^{(j,k)}$. Hence there exists a $j'' < k$ with $O \in \mathbf{O}^{(j'',k)}$. The definition of $O \in \mathbf{O}^{(j'',k)}$ yields an $\overline{O} \in \Psi\left(a^{(j'')}\right)$ such that $(\overline{O}, O) \in \mathcal{E}_{\mathcal{U}}$. We apply $\Rightarrow$ of the induction hypothesis to j''; thus we obtain a Bellman path P from $(O_*, a^{(0)})$ to $\left(\overline{O}, a^{(j'')}\right)$. Assertion (5) says that $P \oplus (\overline{O}, O)$ is the desired Bellman path from $(O_*, a^{(0)})$ to $(O, a^{(k)})$.

<u>*Proof to* " $\Longleftarrow$ "</u>: Here our argumentation is similar to the last part of " $\Rightarrow$ ":

We assume that there exist a Bellman path from $(O_*, a^{(0)})$ to $(O, a^{(k)})$; more precisely, we assume that there exist

- a path $P := (O_0, \ldots, O_r = O)$ and
- objects $a_0 = a^{(0)}, a_1, \ldots, a_r = a^{(k)}$

such that $O_\rho \in \overline{\Psi}(a_\rho)$, $\rho = 0, \ldots, r$; in particular,

(11) $\quad O = O_r \in \overline{\Psi}(a_r) = \overline{\Psi}(a^{(k)})$.

Note now that $O_* \neq O$ because $(O, a^{(k)}) \overset{(11)}{\in} R$ and $(O_*, a^{(k)}) \notin R$ (see *Condition IV)*). Consequently, $O_0 = O_* \neq O = O_r$ so that $r > 0$ and the index $r-1$ really occurs.

Let j' such that $a_{r-1} = a^{(j')}$. Then the following assertions are true:

(12) $\quad (O_{r-1}, a_{r-1}) = \left(O_{r-1}, a^{(j')}\right) \in R$

(13) $\quad (O_r, a^{(k)}) \in R$.

(14) $\quad (O_{r-1}, O_r) = (O_{r-1}, O)$ is an arc of $\mathcal{G}_{\mathcal{U}}$.

So the strict monotonity of R implies that $j' < k$. We then apply direction " $\Leftarrow$ " of (3) to j' and obtain:

$$(15) \qquad O_{r-1} \in \Psi\left(a^{(j')}\right).$$

Moreover, we recall the definition of X for $O := O_r$ and $\overline{O} := O_{r-1}$; then (14) and (13) imply that

(16) $j' \in X$ in the k-th iteration.

These facts are used to complete the proof of (3): Assertions (12) and (13) and the optimality of $O = O_r$ yield $C(O) = \mu\left(O_{r-1}, a^{(k)}\right)$; this and Fact (15) imply that O is put in $\mathbf{O}^{(j',k)}$ in Step 2.0 of the $(j'+1)$st iteration. The consequence of this and of (16) is that $O \in \mathcal{M}$ in Step 2.2 of the kth iteration. Note that $\mathcal{M} \subseteq R^{-1}\left[a^{(k)}\right]$ because $\mathbf{O}^{(j,k)} \subseteq R^{-1}\left[a^{(j,k)}\right]$ for all j. This and the optimality of $O = O_r$ within $R^{-1}\left[a^{(k)}\right]$ implies that $O \in \Psi\left(a^{(k)}\right)$ in Step 2.3.

Now Fact (3) is proven. In order to show Fact (1) we consider an $O \in \Psi(a^{(k)})$. Then " $\Leftarrow$ " of (3) yields a path $(O_0, \ldots, O_r = O)$ and the objects $a_0, \ldots, a_r = a^{(k)}$ such that $O_\rho \in \overline{\Psi}(a_\rho)$ for all ρ. Consequently, $O = O_r \in \overline{\Psi}(a_r) = \overline{\Psi}(a^{(k)})$. This means that indeed $\Psi(a^{(k)}) \subseteq \overline{\Psi}(a^{(k)})$, and Fact (1) is proven.

Fact (2) is shown with the help of Assertion (3). It says that for the given $a^{(k)}$ there exist a path $(O_0, \ldots, O_r)$ and r objects $a_0 = a^{(0)}, \ldots, a_r = a^{(k)}$ with $O_\rho \in \overline{\Psi}(a_\rho)$ for all ρ. But then " $\Leftarrow$ " of Fact (3) says that O_ρ must be an element of $\Psi(a_\rho)$ so that this set is not empty; in particular, this is true for $\rho = r$ so that $\Psi(a_r) = \Psi\left(a^{(k)}\right) \neq \emptyset$.

Now Facts (1), (2) and (3) are proven. Assertion (2) means that Step 3 finds an element $\psi(a^{(k)})$ for all $k = 0, \ldots, n$, and Fact (1) says that $\psi(a^{(k)})$ in $\overline{\Psi}(a^{(k)})$. ■

4.4 The Complexity of DP$_2$

The next results are about the time consumed by DP$_2$.

Theorem 4.3. We define $T_i^{(k)}$ as the time of computation DP$_2$ consumes in Step 2.i of the kth iteration ($i = 0, \ldots, 4,\ k = 1, \ldots, n$). Then the time T consumed by DP$_2$ can be estimated as follows:

a) $T \in \mathcal{O}\left(\sum_{k=1}^{n}\left(T_0^{(k)}+T_1^{(k)}+T_2^{(k)}+T_3^{(k)}\right)\right)$.

b) $T \in \mathcal{O}\left(\sum_{0\le i_1<i_2\le n}\left|\mathcal{N}_{\mathcal{U}}\left(\Psi\left(a^{(i_1)}\right)\right)\cap R^{-1}\left[a^{(i_2)}\right]\right|\right)$.

Proof:

Part a) is almost trivial because only the steps 2.0 – 2.3 cause a substantial amount of computation time.

We next consider Part b) and start with estimating $T_0^{(k)}$. For this we assume that for every $O \in \mathcal{N}_{\mathcal{U}}(\psi(a^{(k)}))$ the set $R[O]$ is available. Then we can give the following description of Step 2.0, which is based on the fact that all elements of $\mathbf{O}^{(k-1,l)}$ have the same costs.

For all $O \in \mathcal{N}_{\mathcal{U}}\left(\Psi\left(a^{(k-1)}\right)\right)$
  For all l with $a^{(l)} \in R[O]$ do
    IF $\mathbf{O}^{(k-1,l)} = \emptyset$ THEN $\mathbf{O}^{(k-1,l)} := \{O\}$;
    IF $\mathbf{O}^{(k-1,l)} \neq \emptyset$ THEN
      Choose an arbitrary $O' \in \mathbf{O}^{(k-1,l)}$;
      IF $C(O') > C(O)$ THEN $\mathbf{O}^{(k-1,l)} := \{O\}$;
      IF $C(O') = C(O)$ THEN $\mathbf{O}^{(k-1,l)} := \mathbf{O}^{(k-1,l)} \cup \{O\}$.

This procedure takes a time of

$$T_0^{(k)} \in \mathcal{O}\left(\sum_{O\in\mathcal{N}_{\mathcal{U}}\left(\Psi\left(a^{(k-1)}\right)\right)}\left|R[O]\right|\right).$$

To estimate this sum we observe that

$$\sum_{O\in\mathcal{N}_{\mathcal{U}}(\Psi(a^{(k-1)}))}\left|R[O]\right| = \left|\left\{(O,l)\,\middle|\, O\in\mathcal{N}_{\mathcal{U}}\left(\Psi\left(a^{(k-1)}\right)\right),\ l>k-1,\ (O,a^{(l)})\in R\right\}\right|$$
$$= \sum_{l>k-1}\left|\mathcal{N}_{\mathcal{U}}(\Psi(a^{(k-1)})\cap R^{-1}[a^{(l)}]\right|.$$

This implies that

$$T_0^{(k)} \in \mathcal{O}\left(\sum_{l>k-1}\left|\mathcal{N}_{\mathcal{U}}\left(\Psi\left(a^{(k-1)}\right)\right)\cap R^{-1}\left[a^{(l)}\right]\right|\right).$$

We next find an upper bound to $T_1^{(k)}$. Recalling *Remark 4.1 (iii)* we see that X can be computed in $|X|$ time units by collecting all j with $\mathbf{O}^{(j,k)} \neq \emptyset$. This means that each $j = 0,\ldots,k-1$ requires $t(j,k) = 1$ time unit if $\mathbf{O}^{(j,k)} \neq \emptyset$, and j costs $t(j,k) = 0$ time units if $\mathbf{O}^{(j,k)} = \emptyset$. Consequently, $t(j,k) \le \left|\mathbf{O}^{(j,k)}\right|$ for *all* $j = 0,\ldots,k-1$. Moreover,

$$(1)\quad (\forall\, j=0,\ldots,k-1)\quad \mathbf{O}^{(j,k)} \subseteq \mathcal{N}_{\mathcal{U}}\left(\Psi\left(a^{(j)}\right)\right)\cap R^{-1}[a^{(k)}].$$

This implies that

$$T_1^{(k)} = \sum_{j=0}^{k-1} t(j,k) \le \sum_{j=0}^{k-1}\left|\mathbf{O}^{(j,k)}\right| \overset{(1)}{\le} \sum_{j=0}^{k-1}\left|\mathcal{N}_{\mathcal{U}}\left(\Psi\left(a^{(j)}\right)\right)\cap R^{-1}\left[a^{(k)}\right]\right|.$$

We next consider $T_2^{(k)}$; for this we assume that $\mathcal{M}$ is represented as a list of all elements $O \in \mathbf{O}^{(j,k)}$ where $j \in X$. (It is not bad if an object $O \in \mathbf{O}^{(j',k)} \cap \mathbf{O}^{(j'',k)}$ occurs more than once.) This description of $\mathcal{M}$ can be obtained by traversing all sets $\mathbf{O}^{(j,k)} \neq \emptyset$, which can be done in the following time:

$$T_2^{(k)} = \sum_{j=0}^{k-1}\left|\mathbf{O}^{(j,k)}\right| \overset{(1)}{\le} \sum_{j=0}^{k-1}\left|\mathcal{N}_{\mathcal{U}}\left(\Psi\left(a^{(j)}\right)\right)\cap R^{-1}\left[a^{(k)}\right]\right|.$$

In order to calculate $T_k^{(3)}$ we observe that the list generated in Step 2.2 has $\sum_{j=0}^{k-1} \left|\mathbf{O}^{(j,k)}\right|$ elements. Hence Step 2.3 takes the following time of computation:

$$T_3^{(k)} \;=\; \sum_{j=0}^{k-1} \left|\mathbf{O}^{(j,k)}\right| \;\overset{(1)}{\leq}\; \sum_{j=0}^{k-1} \left|\mathcal{N}_{\mathcal{U}}\left(\Psi\left(a^{(j)}\right)\right) \cap R^{-1}\left[a^{(k)}\right]\right| .$$

Altogether we see that the time T consumed by DP_2 has the following property:

$$T \;\in\; \mathcal{O}\left(\sum_{k=1}^{n} \sum_{l>k-1} \left|\mathcal{N}_{\mathcal{U}}(\Psi\left(a^{(k-1)}\right) \cap R^{-1}[a^{(l)}]\right| + \sum_{k=1}^{n}\sum_{j=0}^{k-1} \left|\mathcal{N}_{\mathcal{U}}\left(\Psi\left(a^{(j)}\right)\right) \cap R^{-1}[a^{(k)}]\right|\right) .$$

Let $i_1 := l$, $i_2 := k$ in the first sum, and let $i_1 := j$, $i_2 := k$ in the second. Then we see that

$$T \;\in\; \mathcal{O}\left(\sum_{1 \,\leq\, i_1 \,<\, i_2 \,\leq\, n} \left|\mathcal{N}_{\mathcal{U}}(\Psi\left(a^{(i_1)}\right) \cap R^{-1}[a^{(i_2)}]\right|\right) .$$

At the end of this proof we should have a look at the *data structure* that should be used to store the sets $\mathbf{O}^{(i_1,i_2)}, 0 \leq i_1 < i_2 \leq n$. The computation of $T_0^{(k)}$ is based on the assumption that for $i_1 = k-1$ and for all $i_2 > k-1$ the following operations can be executed in 1 time unit:

- testing whether $\mathbf{O}^{(i_1,i_2)}$ is empty,
- choosing an element $O' \in \mathbf{O}^{(i_1,i_2)}$ if $\mathbf{O}^{(i_1,i_2)} \neq \emptyset$,
- replacing $\mathbf{O}^{(i_1,i_2)}$ by $\{O\}$,
- adding a new element to $\mathbf{O}^{(i_1,i_2)}$.

Moreover, when estimating $T_1^{(k)}$ and $T_2^{(k)}$ we assume that for $i_2 = k$ the sets $\mathbf{O}^{(i_1,i_2)} \neq \emptyset$ $(i_1 = 0,\ldots,i_2)$ can directly be found without considering the sets $\mathbf{O}^{(i_1,i_2)} = \emptyset$. In addition, scanning all elements of a given set $\mathbf{O}^{(i_1,i_2)} \neq \emptyset$ must be possible in $|\mathbf{O}^{(i_1,i_2)}|$ steps.

This behaviour can be effected by an array $\mathbf{Z}(i_1,i_2)$ where each element $z(i_1,i_2)$ is a record of three components: $z(i_1,i_2).1$ is equal to nil if $\mathbf{O}^{(i_1,i_2)} = \emptyset$; otherwise $z(i_1,i_2).1$ is a linear list with all elements of $\mathbf{O}^{(i_1,i_2)}$. — If $z(i_1,i_2).1 \neq \text{nil}$ then $z(i_1,i_2).2 = i_1(<)$ and $z(i_1,i_2).3 = i_1(>)$ where $i_1(<)$ and $i_1(>)$ is the next index $i_1' < i_1$ and $i_1' > i_1$ for that $z(i_1',i_2).1 \neq \text{nil}$, respectively. . ■

Remark 4.4. Recall *Example 1.1.* Then the time of computation consumed by DP_2 can be bounded as follows:

a) $T \;\in\; \mathcal{O}\left(\sum_{i=0}^{n-1}\left|\Psi\left(a^{(i)}\right)\right| \cdot \left|N\left(a^{(i)}\right)\right|\right) \;\subseteq\; \mathcal{O}\left(\sum_{i=0}^{n-1}\left|\overline{\Psi}\left(a^{(i)}\right)\right| \cdot \left|N\left(a^{(i)}\right)\right|\right) .$

b) $T \;\in\; \mathcal{O}\left(\max_{i=0,\ldots,n-1} \left|\Psi\left(a^{(i)}\right)\right| \cdot \left|E\right|\right) \subseteq \mathcal{O}\left(\max_{i=0,\ldots,n-1} \left|\overline{\Psi}\left(a^{(i)}\right)\right| \cdot \left|E\right|\right) .$

In particular, $T \in \mathcal{O}(|E|)$ if $\left|\overline{\Psi}\left(a^{(i_1)}\right)\right| = 1$ for all i_1; this condition means that the minimal path from $a^{(0)}$ to $a^{(i_1)}$ is unique for all i_1.

Proof: It is easy to see that for all $0 \leq i_1 < i_2 \leq n$ the following is true:

$$(1) \quad \mathcal{N}_{\mathcal{U}}\left(\Psi\left(a^{(i_1)}\right)\right) \cap R^{-1}\left[a^{(i_2)}\right] \;=\; \begin{cases} \left\{P \oplus \left(a^{(i_1)},a^{(i_2)}\right) \,\middle|\, P \in \Psi\left(a^{(i_1)}\right)\right\} & \textbf{if} \quad \left(a^{(i_1)},a^{(i_2)}\right) \in E\,, \\ \emptyset & \textbf{if} \quad \left(a^{(i_1)},a^{(i_2)}\right) \notin E\,. \end{cases}$$

Consequently, for all $i_1 = 0, \ldots, n-1$,

$$(2) \quad B(i_1) \ := \ \bigcup_{i_2=i_1+1}^{n} \mathcal{N}_{\mathcal{U}}\left(\Psi\left(a^{(i_1)}\right)\right) \cap R^{-1}\left[a^{(i_2)}\right] \\ \left\{P \oplus \left(a^{(i_1)}, a^{(i_2)}\right) \,\middle|\, P \in \Psi\left(a^{(i_1)}\right) \ , \ i_2 > i_2 \ , \ \left(a^{(i_1)}, a^{(i_2)} \in E\right) \right\} \ .$$

So we can define the function $\lambda : B(i_1) \to \Psi\left(a^{(i_1)}\right) \times N\left(a^{(i_1)}\right)$ by $\lambda\left(P \oplus \left(a^{(i_1)}, a^{(i_2)}\right)\right) := \left(P, a^{(i_2)}\right)$ for all paths $P \oplus \left(a^{(i_1)}, a^{(i_2)}\right) \in B(i_1)$. Obviously, λ is bijective; in particular, $\lambda^{-1}\left(P, a^{(i_2)}\right) = P \oplus \left(a^{(i_1)}, a^{(i_2)}\right)$ for all $P \in \Psi\left(a^{(i_1)}\right)$ and $a^{(i_2)} \in N\left(a^{(i_1)}\right)$. So we can conclude that

$$(3) \quad (\forall\, i_1 = 0, \ldots, n-1) \quad |B(i_1)| \ := \ \left|\Psi\left(a^{(i_1)}\right)\right| \cdot \left|N\left(a^{(i_1)}\right)\right|$$

We next observe that the sets $R^{-1}\left[a^{(i_2)}\right]$, $i_2 = 0, \ldots, n$ are pairwise disjoint because different sets $R^{-1}\left[a^{(i_2)}\right]$ and $R^{-1}\left[a^{(i_2')}\right]$ with $i_2 \neq i_2'$ contain paths with different end nodes $a^{(i_2)} \neq a^{(i_2')}$. Of course, the same is true for the sets $\mathcal{N}_{\mathcal{U}}\left(\Psi\left(a^{(i_1)}\right)\right) \cap R^{-1}\left[a^{(i_2)}\right]$, $i_2 = 0, \ldots, n$. This implies that

$$(4) \quad (\forall\, i_1 = 0, \ldots, n-1) \quad \sum_{i_2=i_1+1}^{n} \left|\mathcal{N}_{\mathcal{U}}\left(\Psi\left(a^{(i_1)}\right)\right) \cap R^{-1}\left[a^{(i_2)}\right]\right| \overset{(2)}{=} |B(i_1)| \overset{(3)}{=} \left|\Psi\left(a^{(i_1)}\right)\right| \cdot \left|N\left(a^{(i_1)}\right)\right|$$

So we obtain

$$(5) \quad \sum_{0 \le i_1 < i_2 \le n} \left|\mathcal{N}_{\mathcal{U}}\left(\Psi\left(a^{(i_1)}\right)\right) \cap R^{-1}\left[a^{(i_2)}\right]\right| \ = \ \sum_{i_1=0}^{n-1} \sum_{i_2=i_1+1}^{n} \left|\mathcal{N}_{\mathcal{U}}\left(\Psi\left(a^{(i_1)}\right)\right) \cap R^{-1}\left[a^{(i_2)}\right]\right| \overset{(4)}{=} \\ \sum_{i_1=0}^{n-1} \left|\Psi\left(a^{(i_1)}\right)\right| \cdot \left|N\left(a^{(i_1)}\right)\right| \ .$$

Moreover, we recall Fact (1) in the proof to *Theorem 4.2*:

$$(6) \ \Psi(a^{(i_1)}) \subseteq \overline{\Psi}(a^{(i_1)}) \,, \ \ i_1 = 0, \ldots, n.$$

With the help of these facts we obtain

$$(7) \quad \sum_{0 \le i_1 < i_2 \le n} \left|\mathcal{N}_{\mathcal{U}}\left(\Psi\left(a^{(i_1)}\right)\right) \cap R^{-1}\left[a^{(i_2)}\right]\right| \overset{(5)}{\le} \sum_{i_1=0}^{n-1} \left|\Psi\left(a^{(i_1)}\right)\right| \cdot \left|N\left(a^{(i_1)}\right)\right| \\ \overset{(6)}{\le} \sum_{i_1=0}^{n-1} \left|\overline{\Psi}\left(a^{(i_1)}\right)\right| \cdot \left|N\left(a^{(i_1)}\right)\right| \ = \ \sum_{i=0}^{n-1} \left|\overline{\Psi}\left(a^{(i)}\right)\right| \cdot \left|N\left(a^{(i)}\right)\right| \ .$$

Combining this fact with *Theorem 4.3 b)* we obtain *Assertion a)* of this Remark. In order to prove *Part b)* we observe that

$$\sum_{0 \le i_1 < i_2 \le n} \left|\mathcal{N}_{\mathcal{U}}\left(\Psi\left(a^{(i_1)}\right)\right) \cap R^{-1}\left[a^{(i_2)}\right]\right| \overset{(5)}{\le} \sum_{i_1=0}^{n-1} \left|\Psi\left(a^{(i_1)}\right)\right| \cdot \left|N\left(a^{(i_1)}\right)\right| \\ \le \sum_{i_1=0}^{n-1} \left(\max_{i=0,\ldots,n-1} \left|\Psi\left(a^{(i)}\right)\right|\right) \cdot \left|N\left(a^{(i_1)}\right)\right| \ = \ \left(\max_{i=0,\ldots,n-1} \left|\Psi\left(a^{(i)}\right)\right|\right) \cdot \sum_{i_1=0}^{n-1} \left|N\left(a^{(i_1)}\right)\right| \\ = \ \left(\max_{i=0,\ldots,n-1} \left|\Psi\left(a^{(i)}\right)\right|\right) \cdot |E| \overset{(6)}{\le} \left(\max_{i=0,\ldots,n-1} \left|\overline{\Psi}\left(a^{(i)}\right)\right|\right) \cdot |E| \ .$$

This result and *Theorem 4.3 b)* yield *Assertion b)*. ∎

5. Concluding Remarks

We have studied the two dynamic programming algorithms DP_1 and DP_2 working in relational decision models. The procedure DP_1 can emulate the Ford-Bellman Algorithm for acyclic graphs and the Greedy Procedure for optimal spanning trees. The algorithm DP_2 is more universal then DP_1 as the conditions for the correctness of DP_2 are less restrictive than those for the correctness of DP_1.

An interesting field for future research is the extension of DP_1, DP_2 or similar procedures to more general classes of graphs $\mathcal{G}_\mathcal{U}$, cost functions C and relations R. For example, it seems to be easy to prove that the *original* DP_1 works correctly if the relation R is a Bellman-Ford ordering, i.e.: For all paths $(O_* = O_0, \ldots, O_n)$ in $\mathcal{G}_\mathcal{U}$ and all objects $a_* = a^{(0)} = a^{(i_0)}, a^{(i_1)}, \ldots, a^{(i_n)}$ the following is true: If $(O_\nu\, ,\, a^{(i_\nu)}) \in R\, ,\ \nu = 0, \ldots, n$ then $i_0 < i_1 < \ldots < i_n$. This property is somewhat weaker than the monotonity of R, and it was already investigated in [4].
So it remains to give an generalized version of DP_1 or DP_2 if

- $\mathcal{G}_\mathcal{U}$ is not acyclic,
- R is not even a Bellman-Ford ordering,
- The underlying relational decision model Ξ does not even have the weak Bellman property.

A further idea is to consider not only the graph $\mathcal{G}_\mathcal{U}$ but also an analogous graph $\mathcal{G}_\mathcal{A} = (\mathcal{A}, \mathcal{E}_\mathcal{A})$. With respect to the relationships between $\mathcal{G}_\mathcal{U}$ and $\mathcal{G}_\mathcal{A}$ we can formulate new properties of R and C, and perhaps we can create new versions of DP_1 and DP_2. For example, it may be sensible to require that the objects $a_0 = a_*, a_1 \ldots, a_m$ in a Bellman path (see *Definition 1.11 d)*) must form a path in $\mathcal{G}_\mathcal{A}$

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Appendix

Here we show the optimal successor property of the relational decision model $\Xi = (\mathcal{U}, \mathcal{A}, R, C, \mathcal{G}_{\mathcal{U}})$, where the cost measure C is defined in *Remark 1.9)*. The proof consists of several steps.

Definition A.1. Given a set A, a number $\rho > 0$ and a function $f : A \to \mathbb{R}$. Then f *has a unique minimum with radius* ρ if there exists an $a^* \in A$ such that $f(a) \geq f(a^*) + \rho$ for all $a \neq a^*$. ■

For every $v \in V$ we choose $P^*(v)$ as a minimum cost path from s to v; let $C^*(v) := C(P^*(v))$.
For all $I \subseteq \{0, \ldots, n\}$ let $a^{(I)} := \bigcup_{i \in I} a^{(i)}$; in particular, $a^{(\emptyset)} = \emptyset$ and $a^{(\{i\})} = a^{(i)}$ for all i.
If $I \neq \emptyset$ then we choose a node $v_*^{(I)} \in a^{(I)}$ such that the value $C^*(v)$, $v \in a^{(I)}$ is minimal for $v = v_*^{(I)}$; in particular, $v_*^{(i)} = v_*^{(\{i\})}$ in the description of *Example 1.2* in *Remark 3.8*

We next discuss the step from $k-1$ of k where $k \in \{1, \ldots, n\}$. For this we define

$$\mathcal{X}_k \ := \ \left\{ j \in \{0, \ldots, n-1\} \,\middle|\, a^{(j)} \times a^{(k)} \neq \emptyset \right\} \ .$$

(This set $X = \mathcal{X}_k$ is constructed in Step 2.2 of DP_1.)

Then we prove the following result:

Theorem A.2. We consider the situation of *Example 1.2*. Given the numbers $\Delta, \delta', \delta'', \delta''' > 0$ such that $\Delta > 2\,\delta''' > 2\,\delta''$.

We make the following assumptions:

(i) For every $v \in a^{(\mathcal{X}_k)}$ there exists an arc from v into $a^{(k)}$, i.e. $N(v) \cap a^{(k)} \neq \emptyset$.
Moreover, we assume the existence of the minimum $h^*(v)$ of all candidates $h(v, w)$ where $w \in a^{(k)}$ is a successor of v. We choose the node $w^*(v) \in a^{(k)}$ such that $h(v, w^*(v)) = h^*(v)$.

(ii) For all $v \in a^{(\mathcal{X}_k)}$ and $w \in N(v) \cap a^{(k)}$ with $w \neq w^*(v)$ the following is true: $h(v, w) \ \geq \ h(v, w^*(v)) + \delta'$.
This means that the restriction of h to $E \cap \left(\{v\} \times a^{(k)}\right)$ has a unique minimum with radius δ'.

(iii) $\left(\forall\, v \in a^{(\mathcal{X}_k)}\right) \left(\forall w \in a^{(k)} \cap N(v)\right) \quad \delta'' \ \leq \ |h(v, w)| \ \leq \ \delta'''$.

(iv) $\left(\forall\, I \subseteq \{0, \ldots, k-1\}\,,\; i \neq \emptyset\right) \quad C^*\big|_{a^{(I)}}$ has a unique minimum with radius Δ.

Then $C^*\big|_{a^{(I)}}$ has a unique minimum with radius Δ' for all nonempty sets $I \subseteq \{0, \ldots, k\}$; the quantity Δ' is defined as

$$\Delta' \ \ := \ \ \min\{\, \delta'\,,\, \delta''\,,\, \Delta - 2 \cdot \delta''' \,\}\,.$$

.

Proof: Given $I \subseteq \{0, \ldots, k\}$. Let $z := v_*^{(I)}$ and $\widetilde{z} \neq v_*^{(I)}$. We assume that $z \in a^{(m)}$ and $\widetilde{z} \in a^{(\widetilde{m})}$ where $m, \widetilde{m} \in I$. If $m = k > 0$, then there exists a predecessor y of z on $P^*(z)$, and we choose ℓ such that $y \in a^{(\ell)}$. If $\widetilde{m} = k > 0$ then there exists a predecessor $\widetilde{y}$ of $\widetilde{z}$ on $P^*(\widetilde{z})$, and we choose $\widetilde{\ell}$ such that $\widetilde{y} \in a^{(\widetilde{\ell})}$.
In our proof we often use the strong Bellman principle for additive cost measures on paths in graphs. It says:

(1) Given $v \in V$ and the path $P^*(v)$. We assume that u is the predecessor of v along $P^*(v)$ and that Q is the prefix of $P^*(v)$ that ends at u.
Then Q is an optimal path s to u is C-optimal for u. This means that

$$C(Q) = C(P^*(u)) = C^*(u) \ \text{ and } \ C^*(v) = C(P^*(v)) = C(P^*(u)) + h(u, v) = C^*(u) + h(u, v)\,.$$

In particular, we obtain the assertions $C^*(z) = C^*(y) + h(y, z)$ and $C^*(\widetilde{z}) = C^*(\widetilde{y}) + h(\widetilde{y}, \widetilde{z})$.

(This Bellman principle is the same as the Bellman condition in *Remark 1.12.c)*, where the RDM Ξ of *Example 1.1* is choosen and the quantifiers are defined as $\mathbf{q}_1 := \mathbf{q}_2 :=$ '$\forall$'.)

We must show that

$$(2) \qquad C^*(\widetilde{z}) \;\geq\; C^*(z) + \Delta'.$$

For this we consider four cases:

CASE 1. $\widetilde{m} < k,\;\; m < k\,.$

Let then $I' := \{\widetilde{m}, m\}$. Then $I' \subseteq I$ implies that $a^{(I')} \subseteq a^{(I)}$ so that $C^*(z) = C^*\left(v_*^{(I)}\right)$ is minimal among all values $C^*(v),\; v \in a^{(I')}$. Moreover, $I' \subseteq \{0, \ldots, k-1\}$ so that *Assumption (iv)* can be applied to I'. Thus we obtain $v_*^{(I')} = v_*^{(I)} = z$ because $C^*\big|_{a^{(I')}}$ takes its minimum both in $v_*^{(I')}$ and in $v_*^{(I)}$. A further application of *Assumption (iv)* to I' yields

$$(\mathrm{A}) \qquad C^*(\widetilde{z}) - C^*(z) \;=\; C^*(\widetilde{z}) - C^*\left(v_*^{(I')}\right) \;\geq\; \Delta\,.$$

CASE 2. $\widetilde{m} = k,\; m < k.$

Let $I' := \{\widetilde{\ell}, m\}$. We first prove that

(3) $\widehat{y} := v_*^{(I')} \in a^{(m)}$.

Suppose that Fact (3) is not true; this situation is illustrated in *Figure A*. Then $v_*^{(I')} \in a^{(\widetilde{\ell})} \backslash a^{(m)}$. This and the fact that $z = v^{(I)}$ is an element of $a^{(m)}$ would imply that

(3.1) $\widehat{y} = v_*^{(I')} \neq v_*^{(I)} = z$ and
$\widehat{y} = v_*^{(I')} \in a^{(I)},\;\; z = v_*^{(I)} \in a^{(I')}$.

Applying *Assumption (iv)* to the set $I' \subseteq \{0, \ldots, k-1\}$ we see that $\widehat{y} = v_*^{(I')}$ is the unique minimum of C^* on $a^{(I')}$ with radius Δ. This and (3.1) imply

(3.2) $\quad C^*(z) - C^*(\widehat{y}) \;\geq\; \Delta.$

Note that $\widetilde{\ell} \in \mathcal{X}_k$ because $(\widetilde{y}, \widetilde{z}) \in E$. This and *Assumption (i)* imply that $\widehat{y} = v_*^{(I')}$ has a successor in $a^{(k)}$; let $\widehat{z} := w^*(\widehat{y})$ $= w^*(v^{(I')})$ and $r' := (\widehat{y}, \widehat{z})$. Then

$C^*(z) - C^*(\widehat{z}) \geq C^*(z) - C(P^*(\widehat{y}) \oplus (\widehat{y}, \widehat{z}))$

$= C(z) - C^*(\widehat{y}) - h(\widehat{y}, \widehat{z}) \overset{(3.2),(iii)}{\geq} \Delta - \delta''' > 0\,.$

This means that $C^*(z)$ is greater than $C^*(\widehat{z})$ where $z = v_*^{(I)}$ and $\widehat{z} = w^*\left(v_*^{(I')}\right)$ are elements of $a^{(k)} = a^{(\widetilde{m})} \subseteq a^{(I)}$.
This is a contradiction to the optimality of $C^*(z)$.
Consequently, (3) is true. We next show that even

(4) $\widehat{y} = v_*^{(I')} = v_*^{(I)} = z.$

For this we assume that $v_*^{(I')} \neq v_*^{(I)}$. We then note that $v_*^{(I)} \in a^{(m)} \subseteq a^{(I')}$. Applying *Assumption (iv)* to $I' \subseteq \{0, \ldots, k-1\}$ we obtain

(4.1) $\quad C^*\left(v^{(I)}\right) \;>\; C^*\left(v^{(I')}\right) + \Delta \;>\; C^*\left(v^{(I')}\right)\,.$

On the other hand, $v_*^{(I')} \overset{(3)}{\in} a^{(m)} \subseteq a^{(I)}$; this and the optimality of $C^*\left(v_*^{(I)}\right)$ implies that $C^*\left(v^{(I')}\right) \geq C^*\left(v^{(I)}\right)$, which is a contradiction to (4.1).
After proving Assertion (4) we next consider two subcases. The first is that $\underline{\widetilde{y} = z}$. Then

$$\text{(B.1)} \quad \begin{aligned} |C^*(\widetilde{z}) - C^*(z)| &\overset{(1)}{=} |C(P^*(\widetilde{y}) \oplus h(\widetilde{y}, \widetilde{z})) - C^*(z)| \\ = |C^*(\widetilde{y}) + h(\widetilde{y}, \widetilde{z}) - C^*(z)| &\overset{\widetilde{y} \equiv z}{=} |h(\widetilde{y}, \widetilde{z})| \overset{(iii)}{\geq} \delta''. \end{aligned}$$

The other subcase is $\underline{\widetilde{y} \neq z}$. We note that $z = v_*^{(I)} \overset{(4)}{=} v_*^{(I')}$ and $\widetilde{y} \in a^{(\widetilde{m})} \subseteq a^{(I')}$. So we can apply *Assumption (iv)* to I', and we obtain for $\widetilde{y} \neq z$:

(5) $\quad |C^*(\widetilde{y}) - C^*(z)| \geq \Delta$.

So we get the following result:

$$\text{(B.2)} \quad \begin{aligned} |C^*(\widetilde{z}) - C^*(z)| &\overset{(1)}{=} |C^*(\widetilde{y}) + h(\widetilde{y}, \widetilde{z}) - C^*(z)| \\ \geq |C^*(\widetilde{y}) - C^*(z)| - |h(\widetilde{y}, \widetilde{z})| &\overset{(5),(iii)}{\geq} \Delta - \delta'''. \end{aligned}$$

<u>*CASE 3:* $\quad \widetilde{m} < k, \quad m = k.$</u>
Let $I' := \{\ell, \widetilde{m}\}$. We then prove

(6) $\widehat{y} := v_*^{(I')} \in a^{(\ell)}$.

If <u>not</u>, then $v_*^{(I')} \neq y$ as $y \in a^{(\ell)}$. Moreover, $y \in a^{(I')}$; hence we can apply *Assumption (iv)* to I' and obtain $C^*(y) \geq C^*\left(v_*^{(I')}\right) + \Delta$; this is used in $(\diamond)$ of the following inequality:

$$C^*(z) - C^*\left(v_*^{(I')}\right) \overset{(1)}{=} C^*(y) + h(y,z) - C^*\left(v_*^{(I')}\right) \overset{(\diamond),(iii)}{\geq} \Delta - \delta'' > 0.$$

But then $C^*(y) > C^*\left(v_*^{(I')}\right)$, and this is a contradiction to the fact that $v_*^{(I')}$ is at least as good as the candidate $y \in a^{(\ell)} \subseteq a^{(I')}$.
Hence (6) is true, and we have almost the situation illustrated in *Figure B*.
We next show that even

(7) $\quad \widehat{y} = v_*^{(I')} = y$.

For this we observe that the arc (y,z) is both in E and in $a^{(\ell)} \times a^{(k)}$ so that $\ell \in \mathcal{X}_k$. This, Fact (6) and *Assumption (i)* yield a node $\widehat{z} := w^*(\widehat{y}) \in a^{(k)}$. If (7) is false we can apply *Assumption (iv)* to $\widehat{y} = v^{(I')}$ and $y \in v^{(I')}$; so we obtain $C^*(y) - C^*(\widehat{y}) \geq \Delta$. This inequalty is applied in $(\diamond)$ of the following assertion:

$$\begin{aligned} &C^*(z) - C^*(\widehat{z}) \overset{(1)}{=} \\ &C^*(y) + h(y,z) - C^*(\widehat{y}) - h(\widehat{y}, \widehat{z}) \\ &\quad \overset{(\diamond),(iii)}{\geq} \Delta - 2 \cdot \delta'' > 0. \end{aligned}$$

This is in conflict with the optimality of z; consequently, (7) must be true.

We again consider two cases. If $\underline{y = \widetilde{z}}$ then

(C.1) $|C^*(\widetilde{z}) - C^*(z)| \overset{(1)}{=} |C^*(\widetilde{z}) - C^*(y) - h(y,z)| \overset{y=\tilde{z}}{=} |h(y,z)| \overset{(iii)}{\geq} \delta''$.

The other case is that $y \neq \widetilde{z}$; then Fact (7) says that $v_*^{(I')} \neq \widetilde{z}$. Moreover, $v_*^{(I')} \in a^{(I')}$ and $\widetilde{z} \in a^{(\widetilde{m})} \subseteq a^{(I')}$. Applying *Assumption (iv)* to I' we obtain

$$(8) \quad C^*\left(v_*^{(I')}\right) - C(\widetilde{z}) \geq \Delta .$$

Consequently,

$$\text{(C.2)} \quad |C^*(\widetilde{z}) - C^*(z)| \overset{(1)}{=} |C^*(\widetilde{z}) - C^*(y) - h(y,z)|$$
$$\geq |C^*(\widetilde{z}) - C^*(y)| - |h(y,z)| \overset{(8),(iii)}{\geq} \Delta - \delta'''.$$

CASE 4. $m = k, \quad \widetilde{m} = k.$

Let $I' := \{\ell, \widetilde{\ell}\}$. We first show that

$$(9) \quad \widehat{y} := v_*^{(I')} = y.$$

For this we observe that each of the nodes $y \in a^{(\ell)}$ and $\widetilde{y} \in a^{(\widetilde{\ell})}$ has a successor in $a^{(k)}$ since $(y,z) \in E$, $(\widetilde{y},\widetilde{z}) \in E$ and $z, \widetilde{z} \in a^{(k)}$. It follows that $a^{(I')} = a^{(\ell)} \cup a^{(\widetilde{\ell})} \subseteq a^{(\mathcal{X}_k)}$. Then *Assumption (i)* yields a successor $\widehat{z} := w^*(\widehat{y}) = w^*\left(v^{(I')}\right)$ of $\widehat{y}$. We assume that (9) is false; this situation is illustrated in *Figure C)*. (In contrast to *Figure C)*, the case $\widetilde{y} \in a^{(\widetilde{\ell})}$ is possible, too, if (9) is false.) We then apply *Assumption (iv)* to the nodes $v_*^{(I')}, y \in a^{(I')}$ and obtain $C^*(y) - C^*\left(v_*^{(I')}\right) \geq \Delta$. This fact is used in the inequality $(\diamond)$:

$$C^*(z) - C^*(\widehat{z}) \overset{(1)}{=}$$
$$C^*(y) + h(y,z) - C^*(\widehat{y}) - h(\widehat{y},\widehat{z})$$
$$\overset{(\diamond)}{\geq} \Delta - |h(y,z)| - |h(\widehat{y},\widehat{z})| \overset{(iii)}{\geq}$$
$$\Delta - 2 \cdot \delta''' > 0 .$$

Hence $C^*(z) > C^*(\widehat{z})$ although $\widehat{z} = w^*(\widehat{y}) \in a^{(k)} \subseteq a^{(I)}$ and $z = v^{(I)}$. This is a contradiction to the optimality of z.

We next distinguish between two cases. Let first $y = \widetilde{y}$. Then

$$\text{(D.1)} \quad C^*(\widetilde{z}) - C^*(z) \overset{(1)}{=} C^*(\widetilde{y}) + h(\widetilde{y},\widetilde{z}) - C^*(y) - h(y,z)$$
$$= h(y,\widetilde{z}) - h(y,z) \overset{(ii)}{\geq} \delta' .$$

The other case is that $y \neq \widetilde{y}$. Applying *Assumption (iv)* to $y \overset{(9)}{=} v_*^{(I')}$ and $\widetilde{y} \in a^{(\widetilde{\ell})} \subseteq a^{(I')}$ yields $C^*(\widetilde{y}) - C^*(y) \geq \Delta$. This fact is used in $(\diamond)$ of the following inequality:

$$\text{(D.2)} \quad |C^*(\widetilde{z}) - C^*(z)| = |C^*(\widetilde{y}) + h(\widetilde{y},\widetilde{z}) - C^*(y) + h(y,z)| \overset{(\diamond)}{\geq} \Delta - \delta''' - \delta''' = \Delta - 2 \cdot \delta''' .$$

The assertion follows from combining the facts (A), (B.1), (B.2), (C.1), (C.2), (D.1) and (D.2). ∎

Conclusion A.3. Given the situation as described in *Remark 1.9)*. Let $N_k := \{0, \ldots, k\}$ for all $0 \leq k \leq n$. Then $C^*\big|_{a^{(N_k)}}$ has a unique minimum with radius $R \cdot \alpha^{3k}$.

The proof is an induction on k. For $k = 0$ the assertion is correct because $a^{(0)}$ only consists of the element s. In the step from $k-1$ to k let

$$\Delta := R \cdot \alpha^{3(k-1)}, \quad \delta' := R \cdot \alpha^{3k}, \quad \delta'' := R \cdot \alpha^{3k-1}, \quad \delta''' := R \cdot \alpha^{3k-2}.$$

Then *Theorem A.2* says that $C^*\big|_{a^{(N_k)}}$ has a unique minimum with radius Δ' where

$$\Delta' \;:=\; \min\{\delta', \delta'', \Delta - 2 \cdot \delta'''\} \;=\; \left\{R \cdot \alpha^{3k},\, R \cdot \alpha^{3k-1},\, R \cdot \left(\alpha^{3(k-1)} - 2 \cdot \alpha^{3k-2}\right)\right\}.$$

Then $\Delta' = R \cdot \alpha^{3k}$ as $\alpha \leq \frac{2}{5}$. ∎

Conclusion A.4. Given the situation in *Remark 1.9)*. Then the RDM Ξ has the optimal successor property.

Proof: Given the sets $x = a^{(j)}$, $y = a^{(k)} \in \mathcal{A}$, and the paths $X_1, X_2 \in R^{-1}[x]$, $Y_1 \in R^{-1}[y]$; this means that $u_i := \omega(X_i) \in a^{(i)}$, $i = 1,2$ and $v_1 := \omega(Y_1) \in a^{(j)}$. We assume that $(X_1, Y_1) \in \mathcal{E}_{\mathcal{U}}$ and that X_2 and Y_1 are optimal for $a^{(j)}$ and $a^{(k)}$, respectively. It must be shown that X_2, as well as X_1, has an optimal successor in $a^{(k)}$.

Let us first consider the case $u_1 = u_2$. Then $(u_2, v_1) = (u_1, v_1) \in E$ because $(X_1, Y_1) \in \mathcal{E}_{\mathcal{U}}$. Let $Y_2 := X_2 \oplus (u_2, v_2)$. We then obtain an assertion where ($\diamond$) is based on the optimality of X_2:

$$C(Y_2) \;=\; C(X_2) \oplus h(u_1, v_1) \;\overset{(\diamond)}{\leq}\; C(X_1) + h(u_1, v_1) \;=\; C(Y_1).$$

This and the optimality of Y_1 implies that Y_2 is optimal, too. Moreover, $Y_2 \in \mathcal{N}_{\mathcal{U}}(X_2)$.

The other situation is that

$$(1) \quad u_1 \neq u_2.$$

We show that this case does not occur; the idea is that Y_1 cannot be optimal if (1) is true.

For this purpose first observe that

$$(2) \qquad (u_1, v_1) \in E$$

because $(X_1, Y_1) \in \mathcal{E}_{\mathcal{U}}$. Consequently, $(u_1, v_1) \in a^{(j)} \times a^{(k)} \neq \emptyset$. Hence $j \in \mathcal{X}_k$, and we can apply Assumption (i) of *Remark 1.9)*. So we find a $v_2 \in a^{(k)}$ with $(u_2, v_2) \in E$. We show that the path $Y_2 := X_2 \oplus (u_2, v_2)$ is better than Y_1; this is the desired contradiction to the optimality of Y_1.

The assumption that X_2 and Y_1 are optimal implies that

$$(3) \qquad \begin{array}{l} C(X_2) = C^*(u_2), \\ C(Y_1) = C^*(v_1). \end{array}$$

Fact (2) together with the strong Bellman Principle for additive cost measures yield:

(4) $\quad C(X_1) = C^*(u_1)$.

We next apply *Conclusion A.3* to N_j. Recalling (1) we obtain

(5) $\quad C^*(u_1) \geq C^*(u_2) + R \cdot \alpha^{3j}$.

Moreover, Assumption (ii) of *Remark 1.9)* says that

(6) $\quad |h(u_i, v_i)| \leq R \cdot \alpha^{3k-2}$, $i = 1, 2$.

Consequently,

$$\begin{array}{l} C(Y_1) - C(Y_2) \;=\; C(X_1) + h(u_1, v_1) - C(X_2) - h(u_2, v_2) \;\overset{(4),(3)}{=} \\ C^*(u_1) + h(u_1, v_1) - C^*(u_2) - h(u_2, v_2) \;\overset{(5),(6)}{\geq}\; R \cdot \alpha^{3j} - 2 \cdot R \cdot \alpha^{3k-2} \\ \overset{k-1 \geq j}{\geq}\; R \cdot \alpha^{3(k-1)} - 2 \cdot R \cdot \alpha^{3k-2} \;=\; R \cdot \alpha^{3k-3} \cdot (1 - 2 \cdot \alpha) \;\overset{\alpha \leq 2/5}{>}\; 0. \end{array}$$

This is a contradiction to the optimality of Y_1. Hence assertion (1) is false. ∎