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Sparse Interpolation from Multiple Derivatives

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Abstract

In this note, we consider the problem of interpolating a sparse function from the values of its multiple derivatives at some given point. We give efficient algorithms for reconstructing sparse Fourier series and sparse polynomials over Sturm-Liouville bases. In both cases, the number of evaluations is linear in the sparsity.

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1 Introduction and background

The problem of interpolating a polynomial that admits a representation with a bounded number (sparsity) of nonvanishing coefficients from its evaluations at a small number of specific points is of great importance in computational algebra. This problem was first studied by Zippel [15] and efficient deterministic algorithms were given by Clausen et al [4, 5] and Grigoriev et al [9] for the case of finite fields, and by Grigoriev and Karpinski [7] and Ben-Or and Tiwari [1] for fields of characteristic zero. Numerous generalizations of the sparse interpolation problem are proposed in the literature and have attracted significant attention (cf. [6], [9]).

Lakshman and Saunders [10] considered the problem of interpolating sparse polynomials in the power basis from the values of its multiple derivatives at some given point. We generalize this setting and consider polynomials that admit a sparse representation in Sturm-Liouville bases as well as finite real Fourier series. We show how to embed these problems in the framework of Grigoriev et al [9] on the interpolation of sparse sums of eigenvalues of operators and use the standard technique of adapting the BCH decoding algorithm to derive efficient deterministic algorithms.

We introduce some general notation. Let $\mathcal{C} = \{\mathcal{F}_n\}_{n\in\mathbb{N}_0}$ be a collection of disjoint R-modules of univariate functions on an integral domain R. Let $\mathcal{F}$ denote the R-module generated by $\mathcal{C}$. Each function $f \in \mathcal{F}$ has a unique representation

$$f(x) = \sum_{n\in I} f_n(x) \tag{1}$$

where $f_n \in \mathcal{F}_n$ and $f_n \not\equiv 0$ for $n \in I$. The set I is called the support of f with respect to $\mathcal{C}$ and is denoted by $\mathrm{supp}(f)$. A function $f \in \mathcal{F}$ is called t-sparse if the cardinality of its support is bounded by t, i.e. $|\mathrm{supp}(f)| \leq t$. Let $\mathrm{supp}(f, a) = \{n \in \mathrm{supp}(f) \,|\, f_n(a) \neq 0\}$ denote the support of f at some point $a \in R$. A point $a \in R$ is called generic if $f_n(a) = 0$ implies $f_n \equiv 0$ for all $f_n \in \mathcal{F}_n$ and $n \in \mathbb{N}_0$, i.e. $\mathrm{supp}(f, a) = \mathrm{supp}(f)$ for all $f \in \mathcal{F}$. Note that there might not exist a generic point for $\mathcal{C}$.

The problem of interpolating a sparse function f from the values of its multiple derivatives can be formulated as follows. Let $f^{(i)}$ denote the i-th derivative of f. Given the information of the sparsity t of f and a point $a \in R$ together with access to the list $\{f^{(i)}(a)\}_{i\in\mathbb{N}_0}$ we have to reconstruct f, that is, we have to find the set $\mathrm{supp}(f)$ and then identify the functions $f_n \in \mathcal{F}_n$ for $n \in \mathrm{supp}(f)$ by using as few as possible values of the list.

In [9], Grigoriev et al considered a related, more general framework and showed how to solve the interpolation problem for sparse (multivariate) sums of eigenfunctions of linear operators. We modify their notation to fit our purpose and reformulate their interpolation result which serves as the main subroutine for the sparse interpolation from multiple derivatives.

Let $\mathcal{C}$ and $\mathcal{F}$ be defined as above and let $\Lambda = \{\lambda_n\}_{n\in\mathbb{N}_0}$ be a set of distinct values from R. We define the linear operator $\mathcal{D} : \mathcal{F} \to \mathcal{F}$ with respect to $\mathcal{C}$ and Λ such that $\mathcal{F}_n$ is the λ_n-eigenspace of $\mathcal{D}$, i.e.

$$\mathcal{F}_n = \{f_n \in \mathcal{F} \,|\, \mathcal{D}f_n = \lambda_n f_n\} \tag{2}$$

for any $n \in \mathbb{N}_0$.

Let $f \in \mathcal{F}$ with $f(x) = \sum_{n\in\mathrm{supp}(f)} f_n(x)$ be a t-sparse function. Note that f can also be considered as a t-sparse sum of eigenfunctions of $\mathcal{D}$. Let $\mathcal{D}^j$ denote the j-fold iteration of $\mathcal{D}$. Due to the linearity of $\mathcal{D}$, we have

$$(\mathcal{D}^j f)(x) = \sum_{n\in\mathrm{supp}(f)} \lambda_n^j f_n(x) \tag{3}$$

for $j \in \mathbb{N}_0$. The results of [9] and [6] imply that f can be reconstructed from the values of $(\mathcal{D}^i f)(a)$ for $i = 0, \ldots, 2t-1$ for some generic point $a \in R$ provided that each $\mathcal{F}_n$ is cyclic. Before we recall the modified result for arbitrary a and unrestricted $\mathcal{F}_n$ we prove a statement about the uniqueness of the sparse function being interpolated.

Lemma 1 Let $f \in \mathcal{F}$ be a t-sparse function and $a \in R$. If $(\mathcal{D}^j f)(a) = 0$ for any $0 \le j < t$ then $\operatorname{supp}(f, a) = \emptyset$.

PROOF. Let I be any finite superset of $\operatorname{supp}(f)$ of cardinality t and let $f(x) = \sum_{n \in I} f_n(x)$ with $f_n \in \mathcal{F}_n$. Then (3) yields the linear system of equations

$$\left(\lambda_n^j\right)_{\substack{n \in I \\ 0 \le j < t}} \cdot \left(f_n(a)\right)_{n \in I} = \left((\mathcal{D}^j f)(a)\right)_{0 \le j < t}$$

where the right-hand side is the all-0 vector. Since Λ consists of distinct elements the left-hand side Vandermonde type matrix is nonsingular, and $f_n(a) = 0$ for $n \in I$. Hence $\operatorname{supp}(f, a) = \emptyset$. $\square$

Corollary 2 Let f and g be t-sparse functions and $a \in R$. Then $(\mathcal{D}^i f)(a) = (\mathcal{D}^i g)(a)$ for $0 \le i < 2t$ implies $\operatorname{supp}(f, a) = \operatorname{supp}(g, a)$ and, furthermore, $f_n(a) = g_n(a)$ for all $n \in \operatorname{supp}(f, a)$.

PROOF. Consider the $2t$-sparse function $f - g$. By Lemma 1, the fact that $\mathcal{D}^i(f - g)(a) = 0$ for $0 \le i < 2t$ implies $\operatorname{supp}(f - g, a) = \emptyset$. This in turn is equivalent to $\operatorname{supp}(f, a) = \operatorname{supp}(g, a)$ and $f_n(a) = g_n(a)$ for all $n \in \operatorname{supp}(f, a)$. $\square$

We use the standard technique of adapting the BCH decoding algorithm to prove the following intermediate interpolation result. For a more general statement cf. [6] and [9].

Theorem 3 Let $f \in \mathcal{F}$ be a t-sparse function and let $a \in R$. Given the values of $(\mathcal{D}^i f)(a)$ for $i = 0, \ldots, 2t-1$, we can compute the list of pairs $(\lambda_n, f_n(a))$ for $n \in \operatorname{supp}(f, a)$.

PROOF. Let I be any finite superset of $\operatorname{supp}(f, a)$ of cardinality t and let $f(a) = \sum_{n \in I} f_n(a)$ where $f_n(a) = 0$ for $n \notin \operatorname{supp}(f, a)$. Let $F_t(a)$ denote the matrix $((\mathcal{D}^{i+j} f)(a))_{0 \le i,j < t}$. We claim that the rank of $F_t(a)$ coincides with $k = |supp(f, a)|$. Note that

$$F_t(a) = \left(\lambda_n^j\right)_{\substack{n \in I \\ 0 \le i < t}} \cdot D_I(a) \cdot \left(\lambda_n^j\right)_{\substack{0 \le i < t \\ n \in I}}$$

where $D_I(a)$ is the diagonal matrix $\operatorname{diag}((f_n(a))_{n \in I})$. Hence, the rank of $F_t(a)$ equals $\operatorname{rank}(D_I(a)) = |\operatorname{supp}(f, a)| = k$ and $F_k(a)$ is nonsingular.

We define the auxiliary polynomial $\xi(z) = \prod\limits_{n \in \operatorname{supp}(f, a)} (z - \lambda_n)$. Let $\xi(z) = \sum\limits_{0 \le i \le k} \xi_i z^i$ with $\xi_k = 1$. Then

$$\sum_{i=0}^{k} \xi_i \, (\mathcal{D}^{i+j} f)(a) \; = \; \sum_{i=0}^{k} \sum_{n \in \operatorname{supp}(f, a)} \xi_i \, \lambda_n^{i+j} \, f_n(a) \; = \; \sum_{n \in \operatorname{supp}(f, a)} f_n(a) \lambda_n^j \left(\sum_{i=0}^{k} \xi_i \, \lambda_n^i \right) \; = \; 0$$

for $0 \le j < k$ and the coefficients of ξ can be determined by solving

$$F_k(a) \cdot \left(\xi_i\right)_{0 \le i < k} = -\left((\mathcal{D}^{k+j} f)(a)\right)_{0 \le j < k}.$$

Determining the roots of ξ yields $\{\lambda_n \mid n \in \operatorname{supp}(f, a)\}$. With (3) we have

$$\left(\lambda_n^j\right)_{\substack{n \in \operatorname{supp}(f, a) \\ 0 \le j < k}} \cdot \left(f_n(a)\right)_{n \in \operatorname{supp}(f, a)} = \left((\mathcal{D}^j f)(a)\right)_{0 \le j < k}$$

which completes the proof. We summerize the steps of the algorithm.

Input: $a \in R$, $t \ge |\operatorname{supp}(f)|$, and $(\mathcal{D}^i f)(a)$, $i = 0, \ldots, 2t - 1$.

Output: List of pairs $(\lambda_n, f_n(a))$ for $n \in \operatorname{supp}(f, a)$.

Step 1: Let k be the rank of $((\mathcal{D}^{i+j}f)(a))_{0\le i,j<t}$. Then $k = |\operatorname{supp}(f,a)|$.

Step 2: Solve the linear system of equations

$$\Big((\mathcal{D}^{i+j}f)(a)\Big)_{0\le i,j<k}\cdot\Big(\xi_i\Big)_{0\le i<k} = -\Big((\mathcal{D}^{k+j}f)(a)\Big)_{0\le j<k}.$$

Step 3: Find the roots of the polynomial $\xi(z) = \sum_{i=0}^{k-1}\xi_i z^i + z^k$. These roots are $\{\lambda_n \mid n \in \operatorname{supp}(f,a)\}$.

Step 4: Solve the linear system of equations

$$\Big(\lambda_n^j\Big)_{\substack{n\in\operatorname{supp}(f,a)\\ 0\le j<k}}\cdot\Big(f_n(a)\Big)_{n\in\operatorname{supp}(f,a)} = \Big((\mathcal{D}^j f)(a)\Big)_{0\le j<k}.$$

□

2 Interpolation of sparse Fourier series

As an example of the interpolation problem of sparse sums of eigenfunctions of operators, Grigoriev et al considered the problem of interpolating finite multivariate Fourier series over the complex numbers from the values of their partial derivatives [9]. In this section, we consider the corresponding interpolation problem for finite univariate Fourier series over the real numbers. In this case, the corresponding R-modules $\mathcal{F}_n$ are not cyclic, nor does a generic point $a \in R$ exit.

Let $\mathcal{F}_n = \{a_n \cos nx + b_n \sin nx \mid a_n, b_n \in \mathbb{R}\}$ for $n \in \mathbb{N}$ and $\mathcal{F}_0 = \{a_0 \mid a_0 \in \mathbb{R}\}$. It is convenient to set $b_0 = 0$. Then $\mathcal{F}$ is the set of finite univariate Fourier series over the reals.

A Fourier series f is called t-sparse if f admits a representation

$$f(x) = \sum_{n\in\operatorname{supp}(f)} (a_n \cos nx + b_n \sin nx)$$

with $|\operatorname{supp}(f)| \le t$. The list $\{(n, a_n, b_n) \mid n \in \operatorname{supp}(f)\}$ is called sparse representation of f. Note that a t-sparse Fourier series has at most $2t$ nonvanishing Fourier coefficients.

Theorem 4 Let f be a t-sparse Fourier series. The knowledge of the values of $f^{(i)}(a)$ for $i = 0, \dots, 4t-1$ at some arbitrary given point $a \in \mathbb{R}$ enables us to uniquely compute the sparse representation of f.

PROOF. Note that $f_n^{(2)}(x) = -n^2 f_n(x)$ for $f_n \in \mathcal{F}_n$, $n \in \mathbb{N}_0$. Hence, we define $\lambda_n = -n^2$ and have $\mathcal{D}(f) = f^{(2)}$ for $f \in \mathcal{F}$. Let

$$f(x) = \sum_{n\in\operatorname{supp}(f)} f_n(x)$$

with $|\operatorname{supp}(f)| \le t$ and $f_n(x) = a_n \cos nx + b_n \sin nx$. Using Theorem 3, we compute the lists

$$\{(-n^2, f_n(a)) \mid n \in \operatorname{supp}(f,a)\} \quad \text{and} \quad \{(-n^2, f_n'(a)) \mid n \in \operatorname{supp}(f',a)\}$$

from the values $(\mathcal{D}^i f)(a) = f^{(2i)}(a)$, $i = 0, \dots, 2t-1$ and $(\mathcal{D}^i f')(a) = f^{(2i+1)}(a)$, $i = 0, \dots, 2t-1$. Note that

$$\begin{pmatrix} \cos na & \sin na \\ -n\sin na & n\cos na \end{pmatrix}\cdot\begin{pmatrix} a_n \\ b_n \end{pmatrix} = \begin{pmatrix} f_n(a) \\ f_n'(a) \end{pmatrix} \tag{4}$$

for $n \in \mathbb{N}$ and $a_0 = f_0(a)$. Since the left-hand matrix is nonsingular for $n \in \mathbb{N}$ and any point $a \in \mathbb{R}$, $f_n(a) = f'_n(a) = 0$ implies $a_n = b_n = 0$. Hence

$$\text{supp}(f) = \text{supp}(f, a) \cup \text{supp}(f', a),$$

and the coefficients a_n, b_n of $f_n(x)$ for $n \in \text{supp}(f)$ are easily determined by solving (4). The uniqueness of the reconstruction follows from Corollary 2 and the uniqueness of the solution of (4). □

For the analysis of the above algorithm we assume that the evaluation of $\cos(x)$ and $\sin(x)$ takes one field operation for any $x \in \mathbb{R}$.

Lemma 5 The sparse Fourier series interpolation algorithm takes $O(t^2 + t \log d)$ field operations to interpolate a t-sparse Fourier series f where d is the (à-priori unknown) largest element in $\text{supp}(f)$.

PROOF. The algorithm suggested in Theorem 3 is invoked twice. Step 1 and 2 can be performed in $O(t^2)$ field operations using the Berlekamp-Massey algorithm [2] for Toeplitz matrices. By using Zippel's algorithm [16] for the inversion of Vandermonde matrices, Step 4 also takes $O(t^2)$ steps. In Step 3 we have to find the integer roots of the monic polynomial $\xi \in \mathbb{Z}[x]$ whose roots are bounded by d^2 in absolute value. This can be performed in $O(t^2 + t \log d)$ field operations using Hensel lifting (cf. [13, 1, 10]). Under the assumption that the evaluation of $\cos(x)$ and $\sin(x)$ takes one field operation the determination (4) of the nonvanishing Fourier coefficients of f takes $O(t)$ steps. □

3 Interpolation in Sturm-Liouville bases

In [10], Lakshman and Saunders solved the interpolation problem from multiple derivatives for the case of univariate polynomials that admit a sparse representation in the standard power basis. In this section, we derive efficient algorithms for the more general case of sparse polynomials over Sturm-Liouville bases which include the important classical orthogonal bases. For the case of the power basis our algorithm coincides with the algorithm given by Lakshman and Saunders, however, we show that their intermediate restriction on the minimal degree of the occurring monomials is not necessary.

The notion of a Sturm-Liouville basis was introduced by Bochner in [3]. Let K denote the field of complex or real numbers and let $\mathcal{P} = \{P_n\}_{n \in \mathbb{N}_0}$ be a sequence of univariate polynomials over K in which the degree of P_n is exactly n for $n \in \mathbb{N}_0$. The set $\mathcal{P}$ is called a Sturm-Liouville polynomial system iff there exists a set $\Lambda = \{\lambda_n\}_{n \in \mathbb{N}_0}$, $\lambda_n \in K$ of parameter values such that for each $n \in \mathbb{N}_0$ the polynomial P_n is a solution of a second order linear differential equation of the Sturm-Liouville type

$$\sigma(x)\, y''(x) + \tau(x)\, y'(x) + \lambda_n\, y(x) = 0 \tag{5}$$

where $\sigma(x)$ and $\tau(x)$ are fixed polynomial over K of degree 2 and 1 resp., both independent of n, and λ_n is independent of x. Clearly, $\mathcal{P}$ forms a basis for $K[x]$.

Bochner characterized all Sturm-Liouville bases. Essentially, these are, modulo linear transformations, the power basis, the generalized Jacobi, Laguerre and Hermite bases as well as a generalized Bessel basis. Furthermore, he showed that the parameter values are pairwise distinct and of the form $\lambda_n = \alpha\, n\, (n-1) + \beta\, n$ for some constants $\alpha, \beta \in K$.

A polynomial $f \in K[z]$ is called t-sparse with respect to some Sturm-Liouville basis $\mathcal{P}$ iff f admits a representation

$$f(x) = \sum_{n \in \text{supp}(f)} c_n\, P_n(x)$$

with $0 \neq c_n \in K$ and $|\operatorname{supp}(f)| \leq t$. Using our notation we set $\mathcal{F}_n = \{c\, P_n \mid c \in K\}$ and define the linear operator $\mathcal{D}$ with respect to the eigenvalues $\{\lambda_n\}_{n \in \mathbb{N}_0}$ given by (5). Then

$$(\mathcal{D}^j f)(x) = \sum_{n \in \operatorname{supp}(f)} \lambda_n^j\, c_n\, P_n(x).$$

Lemma 6 Let $f \in K[x]$. Given $a \in K$ and the values of $f^{(i)}(a)$ for $i = 0, \ldots, 2l$, we can determine the values of $(\mathcal{D}^j f)(a)$ for $j = 0, \ldots, l$ in $O(l^2)$ field operations.

PROOF. For each $j = 1, \ldots, l$ we show how to compute $(\mathcal{D}^j f^{(i)})(a)$ for $i = 0, \ldots, 2(l-j)$ from the values of $(\mathcal{D}^{j-1} f^{(i)})(a)$ for $i = 0, \ldots, 2(l-j)+2$ in $O(l)$ field operations. Note that the values of $(\mathcal{D}^0 f^{(i)})(a) = f^{(i)}(a)$ for $i = 0, \ldots, 2l$ are given. Multiplying (5) by λ_n^{j-1} yields

$$\lambda_n^j\, P_n^{(0)}(x) = -\lambda_n^{j-1} \left(\sigma(x)\, P_n^{(2)}(x) + \tau(x)\, P_n^{(1)}(x)\right)$$

for $j \in \mathbb{N}$. Note that $\sigma^{(3)}(x) \equiv 0$ and $\tau^{(2)}(x) \equiv 0$. Hence, by Leibniz' rule, for $i \in \mathbb{N}_0$

$$\lambda_n^j\, P_n^{(i)}(x) = -\sigma(x)\, \lambda_n^{j-1}\, P_n^{(i+2)}(x) - (i\, \sigma'(x) + \tau(x))\, \lambda_n^{j-1}\, P_n^{(i+1)}(x) - \left(\tbinom{i}{2}\, \sigma''(x) + i\, \tau'(x)\right) \lambda_n^{j-1}\, P_n^{(i)}(x).$$

Therefore, for any $f \in K[x]$ and $i \in \mathbb{N}_0$, $j \in \mathbb{N}$ we have

$$\begin{aligned}(\mathcal{D}^j f^{(i)})(a) \quad &= \quad -\sigma(a)\, (\mathcal{D}^{j-1} f^{(i+2)})(a) - (i\, \sigma'(a) + \tau(a))\, (\mathcal{D}^{j-1} f^{(i+1)})(a) \\ &\quad\quad -\left(\tbinom{i}{2}\, \sigma''(a) + i\, \tau'(a)\right) (\mathcal{D}^{j-1} f^{(i)})(a).\end{aligned}$$

Iterating the above procedure for $j = 1, \ldots, l$ completes the proof. □

Theorem 7 Let $\mathcal{P}$ be a Sturm-Liouville basis for $K[x]$ and let $f \in K[x]$ be a t-sparse polynomial over $\mathcal{P}$. The sparse representation of f can be reconstructed from $f^{(i)}(a)$ for $i = 0, \ldots, 4t-2$ at some generic point $a \in K$ in $O(dt \log t)$ field operations where d is the (à-priori unknown) degree of f.

PROOF. Let $f = \sum_{n \in \operatorname{supp}(f)} c_n\, P_n(x)$. Using Lemma 6 we compute $(\mathcal{D}^j f)(a)$ for $j = 0, \ldots, 2t-1$ in $O(t^2)$ field operations. Since $a \in K$ is a generic point for $\mathcal{P}$, Theorem 3 yields the list of pairs $(\lambda_n, c_n\, P_n(a))$ for $n \in \operatorname{supp}(f)$ from which $\operatorname{supp}(f)$ and the coefficients c_n for $n \in \operatorname{supp}(f)$ can be easily determined.

The complexity is dominated by the root finding step which can be done using $O(dt \log t)$ field operations by exhaustive search and the evaluations of $P_n(a)$, $n \in \operatorname{supp}(f)$, which take $O(dt)$ field operations. □

The number of evaluations points can be reduced to $2t$ if the generic point satisfies $\sigma(a) = 0$. These points are $a = \pm 1$ for the Jocobi basis, $a = 0$ for the orthogonal Laguerre basis and $a \neq 0$ for the power basis.

The complexity of the interpolation algorithm can be improved to $O(t^2 + t \log d)$ for the case of the power basis and the Chebyshev basis. In both cases the the eigenvalues are integers, hence, as in the sparse Fourier series interpolation algorithm, we can find the roots of the auxiliary polynomial in $O(t^2 + t \log d)$ field operations. Furthermore, the monomials $P_n(x)$ can be evaluated using binary powering in $O(\log n)$ field operations. To see this for the case of the Chebyshev basis $\{T_n\}_{n \in \mathbb{N}_0}$, note that $T_{n+m}(x) + T_{|n-m|}(x) = 2\, T_n(x)\, T_m(x)$ for $n, m \in \mathbb{N}_0$. Hence

$$T_{2k}(x) = 2\, T_k(x)\, T_k(x) - 1 \qquad T_{2k+1}(x) = 2\, T_{k+1}(x)\, T_k(x) - x$$

and the pair $(T_n(x), T_{n+1}(x))$ can be computed from $(T_{\lfloor n/2 \rfloor}(x), T_{\lfloor n/2 \rfloor + 1}(x))$ using a constant number of field operations.

The restriction to generic points $a \in K$ in Theorem 7 can be relaxed for the classical orthogonal bases (Jacobi, Laguerre and Hermite). One of the first results in the theory of orthogonal polynomials is that the zeros of these polynomials are all real, simple and lie in the interval of orthogonality (cf. [14]). Hence, for any point $a \in K$ and $f \in K[x]$,

$$\mathrm{supp}(f) = \mathrm{supp}(f, a) \cup \mathrm{supp}(f', a)$$

with respect to a classical orthogonal basis. Using Theorem 7 to interpolate $\sum_{n \in \mathrm{supp}(f,a)} c_n\, P_n(x)$ and $\sum_{n \in \mathrm{supp}(f',a)} c_n\, P'_n(x)$, we have

Corollary 8 Let $\mathcal{P}$ be a classical orthogonal basis and let $f \in K[x]$ be a t-sparse polynomial over $\mathcal{P}$. The sparse representation of f can be reconstructed from the values of $f^{(i)}(a)$ for $i = 0, \ldots, 4t-1$ for an arbitrary point $a \in K$.

The results of this section can be generalized to polynomial systems defined by finite difference equations of the Sturm-Liouville type. Let Δ denote the finite difference operator, i.e. $\Delta f(x) = f(x+1) - f(x)$. Instead of (5) we consider the difference equation

$$\sigma(x)\, \Delta^2 y(x) + \tau(x)\, \Delta y(x) + \lambda_n\, y(x+1) = 0, \tag{6}$$

where again $\sigma(x)$ and $\tau(x)$ are fixed polynomial over K of degree 2 and 1 resp., both independent of n, and λ_n is independent of x. A polynomial system $\mathcal{P} = \{P_n\}_{n \in \mathbb{N}_0}$ is called a discrete Sturm-Liouville polynomial system with respect to a set $\Lambda = \{\lambda_n\}_{n \in \mathbb{N}_0}$, $\lambda_n \in K$ of parameter values iff for each $n \in \mathbb{N}_0$, P_n is a solution of (6). A complete characterization of all discrete Sturm-Liouville bases was given by Lancaster [11] and Lesky [12]. The solutions are, modulo linear transformations, the falling factorial powers and the generalized Charlier, Krawtchouk, Meixner and Hahn polynomials. The parameter values are of the form $\lambda_n = \alpha\, n\, (n-1) + \beta\, n$ for some constants $\alpha, \beta \in K$, however, two of them might not be distinct. Let the linear operator $\mathcal{D}$ with respect to $\mathcal{P}$ be defined as above.

Lemma 9 Let $f \in K[x]$. Given $a \in K$ and the values of $\Delta^i f(a - l)$ for $i = 0, \ldots, 2l$, or, alternatively, the values of $f(a+i)$ for $i = -l, \ldots, l$, we can determine the values of $(\mathcal{D}^j f)(a)$ for $j = 0, \ldots, l$ in $O(l^2)$ field operations.

PROOF. Rewrite (6) as $\lambda_n\, y(x) = \rho(x)\, y(x-1) + \mu(x)\, y(x) + \nu(x)\, y(x+1)$ with polynomials ρ, μ, ν of degree at most two. Hence,

$$(\mathcal{D}^j f)(x) = \rho(x)\, (\mathcal{D}^{j-1} f)(x-1) + \mu(x)\, (\mathcal{D}^{j-1} f)(x) + \nu(x)\, (\mathcal{D}^{j-1} f)(x+1)$$

and $(\mathcal{D}^j f)(a+i)$ for $i = -l+j, \ldots, l-j$ can be computed from $(\mathcal{D}^{j-1} f)(a+i)$ for $i = -l+j+1, \ldots, l-j-1$ in $O(l)$ field operations. $\square$

Corollary 10 Let $\mathcal{P}$ be a discrete Sturm-Liouville polynomial system with pairwise distinct parameter values. Let $a \in K$ be a generic point for $\mathcal{P}$. Given $t \in \mathbb{N}_0$, the sparse representation of any t-sparse polynomial $f \in K[x]$ can be uniquely reconstructed from the values of $f(a+i)$ for $i = -2t-1, \ldots, 2t-1$.

If two parameter values, say λ_n and λ_m, of a discrete Sturm-Liouville polynomial system $\mathcal{P}$ are equal, we compute $(\mathcal{D}^j f)(a)$ and $(\mathcal{D}^j f)(a+1)$ for $j = 0, \ldots, 2t-1$ from the $4t$ values of $f(a+i)$ for $i = -2t-1, \ldots, 2t$ for some generic point a such that $P_n(a)\, P_m(a+1) \neq P_n(a+1)\, P_m(a)$ and proceed similar to Theorem 4.

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