Block Korkin–Zolotarev Bases and Successive Minima

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Abstract

A lattice basis $b_1, \ldots, b_m$ is called block Korkin–Zolotarev with block size β if for every β consecutive vectors $b_i, \ldots, b_{i+\beta-1}$ the orthogonal projections of $b_i, \ldots, b_{i+\beta-1}$ in $\mathrm{span}(b_1, \ldots, b_{i-1})^{\perp}$ are reduced in the sense of Hermite and Korkin–Zolotarev. Let λ_i denote the successive minima of lattice L and let $b_1, \ldots, b_m$ be a basis of L that is block Korkin–Zolotarev with block size β. We prove that for $i = 1, \ldots, m$

$$\frac{4}{i+3}\, \gamma_\beta^{-2\frac{i-1}{\beta-1}} \;\leq\; \|b_i\|^2\, \lambda_i^{-2} \;\leq\; \gamma_\beta^{2\frac{m-1}{\beta-1}}\, \frac{i+3}{4}$$

where γ_β is the Hermite constant for dimension β. For block size $\beta = 3$ and odd $m \geq 3$ we show that $\max \|b_1\|^2 \lambda_1^{-2} = \sqrt{\frac{3}{2}}^{m-3}$, where the maximum is taken over all block Korkin–Zolotarev bases of lattices L. We present "critical" block Korkin–Zolotarev bases achieving this maximum.

Using block Korkin–Zolotarev bases we improve Babai's construction of a nearby lattice point. Given a block Korkin–Zolotarev basis with block size β of the lattice L and given a point x in the span of L, a lattice point v can be found in time $\beta^{O(\beta)}$ satisfying $\|x - v\|^2 \leq m\, \gamma_\beta^{\frac{2m}{\beta-1}}\, \min_{u \in L} \|x - u\|^2$. These results also bear improvements for the method of solving integer programming problems via basis reduction.

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1 Introduction

The problem of selecting from all bases for a lattice a canonical basis consisting of short vectors is called *reduction theory*. Classical reduction theory was developed in the language of quadratic forms by LAGRANGE (1773), HERMITE (1850) and KORKIN and ZOLOTAREV (1873) in order to determine the minima of positive definite integral quadratic forms. Recently there has been renewed interest in reduction theory stimulated by a new method in integer programming (H.W. LENSTRA, JR. 1983) and by Lovász' lattice basis reduction algorithm, see A.K. LENSTRA, H.W. LENSTRA JR. and L. LOVÁSZ (1982), which has had various applications. For survey papers and background information see LOVÁSZ (1986) and (1988) and GROETSCHEL, LOVÁSZ, SCHRIJVER (1988).

The quality of a reduced basis $b_1, \ldots, b_m$ can be characterized by the ratios $\|b_i\|^2 \lambda_i^{-2}$, $i = 1, \ldots, m$ where λ_i is the i–th successive minimum of the lattice. These ratios are polynomial in m if the basis is reduced in the sense of HERMITE and KORKIN, ZOLOTAREV whereas $\|b_i\|^2 \lambda_i^{-2}$ may be exponential in m for the L^3–reduced bases introduced by LENSTRA, LENSTRA, LOVÁSZ (1982). From the computational point of view the reduction theory introduced by HERMITE (1850) and by KORKIN and ZOLOTAREV (1873) is the most natural one. On the other hand Korkin–Zolotarev bases are not easy to find for lattices of higher dimensions, see KANNAN (1987) and SCHNORR (1987) for algorithms. This led SCHNORR (1987) to introduce a hierarchy of block Korkin–Zolotarev bases comprising for block size 2 the L^3–reduced bases and for the maximal block size the Korkin–Zolotarev bases.

In this paper we prove new upper and lower bounds for the ratio $\|b_i\|^2 \ \lambda_i^{-2}$ that hold true for block Korkin–Zolotarev bases. For a lattice basis $b_1, \ldots, b_m$ that is block Korkin–Zolotarev with block size β we show for $i = 1, \ldots, m$ that

$$\frac{4}{i+3} \ \gamma_\beta^{-2 \frac{i-1}{\beta-1}} \ \leq \ \|b_i\|^2 \ \lambda_i^{-2} \ \leq \ \gamma_\beta^{2 \frac{m-1}{\beta-1}} \ \frac{i+3}{4}$$

where γ_β is the Hermite constant for dimension β. It is known from SCHNORR (1987) that the base of the exponential bound on $\|b_i\|^2 \ \lambda_i^{-2}$ converges to 1 for $\beta \to \infty$. The new result shows that the base $\gamma_\beta^{\frac{2}{\beta-1}}$ decreases noticeably even as β increases for small values, e.g. $\gamma_\beta^{\frac{2}{\beta-1}} \approx 1.333, 1.260, 1.260, 1.231, 1.226, 1.219, 1.189$ for $\beta = 2, 3, 4, 5, 6, 7, 8$. This explains in theory the experimental results of SCHNORR, EUCHNER (1991) that show a considerably improved success rate in solving subset sum problems using block Korkin–Zolotarev bases of block size $\beta = 10$ and $\beta = 20$ versus $\beta = 2$. According to their experiments almost all subset sum problems of dimension up to 66 can be solved via block Korkin–Zolotarev bases within a couple of hours.

The new bounds on $\|b_i\|^2 \ \lambda_i^{-2}$ are not optimal for block size $\beta \geq 3$. We show that the optimal upper bound for $\beta = 3$ and odd $m \geq 3$ is

$$\|b_1\|^2 \ \lambda_1^{-2} \ \leq \ \sqrt{\frac{3}{2}}^{m-3} \ .$$

The base $\sqrt{\frac{3}{2}} \approx 1.225$ of this exponential bound is smaller than $\gamma_6^{2/5}$, which is the base of the general bound for $\beta = 6$. We present block Korkin–Zolotarev bases $b_1, \ldots, b_m$ with odd

$m \geq 3$ and block size $\beta = 3$ satisfying $\|b_1\|^2 \ \lambda_1^{-2} = \sqrt{\frac{3}{2}}^{m-3}$. These lattice bases are derived from the lattice packing D_3 that has maximal density for dimension 3. We recursively extend D_3 in a "periodic way" to arbitrary dimensions $m \geq 3$.

The new bounds on $\|b_i\|^2 \ \lambda_i^{-2}$ also yield good bounds for the nearest lattice point obtained from block Korkin–Zolotarev bases. Given a block Korkin–Zolotarev basis $b_1, \ldots, b_m$ with block size β and given a point x in the span of the lattice L, a lattice point v can be found in time $\beta^{O(\beta)}$ satisfying

$$\|x - v\|^2 \leq m \, \gamma_\beta^{\frac{2m}{\beta-1}} \min_{u \in L} \|x - u\|^2 \ .$$

This improves and extends to arbitrary β the bound proven by Babai (1986) for $\beta = 2$. Babai shows that $\|x - v\|^2 \leq 2^m \min_{u \in L} \|x - u\|^2$ holds for $\beta = 2$. He poses the problem to improve this bound for $\beta > 2$.

The shortest lattice vector algorithm and the nearby lattice point algorithm are crucial parts of the method to solve integer programming problems via basis reduction (see e.g. Lenstra 1983 and Kannan 1987). Our improved variants for these algorithms also improve the basis reduction method for general integer programming. The power of the improved method has been demonstrated by Schnorr, Euchner (1991) for the particular case of subset sum problems.

We state in Section 2, Theorems 3 and 4 the new lower and upper bounds for the ratios $\|b_i\|^2 \ \lambda_i^{-2}$. We prove these bounds in Section 3. In Section 4 we apply block Korkin–Zolotarev bases to the problem of finding approximate nearest lattice points. In Section 5 we prove that the upper bound $\|b_1\|^2 \leq \sqrt{\frac{3}{2}}^{m-3} \lambda_1^2$ is optimal for block Korkin–Zolotarev bases of block size 3. We present block Korkin–Zolotarev bases for which this bound is tight.

2 Reduced Bases and Successive Minima

Let $\mathbb{R}^n$ be the n–dimensional real vector space with the Euclidean inner product $< , >$ and the Euclidean norm, called the length, $\|y\| = \langle y, y \rangle^{1/2}$. A *lattice* $L \subset \mathbb{R}^n$ is a discrete, additive subgroup of the $\mathbb{R}^n$. Its *rank* is the dimension of the minimal subspace span(L) that contains L. Each lattice L of rank m has a *basis*, i.e. a set $b_1, \ldots, b_m$ of linearly independent vectors that generate L as an abelian group,

$$L = \{t_1 b_1 + \ldots + t_m b_m \,|\, t_1, \ldots, t_m \in \mathbb{Z}\}.$$

Let $L(b_1, \ldots, b_m)$ denote the lattice with basis $b_1, \ldots, b_m$. Its *determinant* $d(L)$ is defined as

$$d(L) = \det[\langle b_i, b_j \rangle]_{1 \leq i,j \leq m}^{1/2}.$$

This does not depend on the choice of the basis. The i–th *successive minimum* $\lambda_i(L)$ of a lattice L (with respect to the Euclidean norm) is the smallest real number r such that there are i linearly independent vectors in L of length at most r.

With an ordered lattice basis $b_1, \ldots, b_m \in \mathbb{R}^n$ we associate the *Gram–Schmidt orthogonalization* $\widehat{b}_1, \ldots, \widehat{b}_m \in \mathbb{R}^n$ which can be computed together with the Gram–Schmidt coefficients $\mu_{i,j} = \langle b_i, \widehat{b}_j \rangle \,/\, \langle \widehat{b}_j, \widehat{b}_j \rangle$ by the recursion

$$\widehat{b}_1 = b_1, \ \widehat{b}_i = b_i - \sum_{j=1}^{i-1} \mu_{i,j} \widehat{b}_j \quad for \quad i = 2, \ldots, m.$$

We have $\mu_{i,i} = 1$ and $\mu_{i,j} = 0$ for $i < j$. From the above equations we have the Gram–Schmidt decomposition

$$[b_1, \ldots, b_m] = [\widehat{b}_1, \ldots, \widehat{b}_m] \, [\mu_{i,j}]_{1 \le i,j \le m}^{\top},$$

where $[b_1, \ldots, b_m]$ denotes the matrix with column vectors $b_1, \ldots, b_m$ and $A^{\top}$ is the transpose of the matrix A.

With an ordered lattice basis $b_1, \ldots, b_m$ of the lattice L we associate the orthogonal projections

$$\pi_i : \mathrm{span}(b_1, \ldots, b_m) \ \mapsto \ \mathrm{span}(b_1, \ldots, b_{i-1})^{\perp} \quad i = 1, \ldots, m \ ,$$

where $\mathrm{span}(b_1, \ldots, b_m)$ denotes the linear space generated by the vectors $b_1, \ldots, b_m$ and $\mathrm{span}(b_1, \ldots, b_{i-1})^{\perp}$ is the orthogonal complement of $\mathrm{span}(b_1, \ldots, b_{i-1})$. We let L_i denote the lattice

$$L_i = \pi_i(L) \ .$$

The lattice L_i has rank $m - i + 1$ and basis $\pi_i(b_i), \ \pi_i(b_{i+1}), \ldots, \pi_i(b_m)$. We have $\pi_i(b_i) = \widehat{b}_i$ and $\pi_i(b_j) = \sum_{k=i}^{m} \mu_{j,k} \widehat{b}_k$.

An ordered basis $b_1, \ldots, b_m$ of the lattice $L \subset \mathbb{R}^n$ is a *Korkin–Zolotarev basis* if it satisfies the conditions

$$|\mu_{i,j}| \ \le \ 1/2 \qquad \text{for} \quad 1 \le j < i \le m, \tag{1}$$

$$\|\widehat{b}_i\| \ = \ \lambda_1(L_i) \quad \text{for} \quad i = 1, \ldots, m. \tag{2}$$

This definition is equivalent to the one given, in the language of quadratic forms, by Hermite (1850) in his second letter to Jacobi and by Korkin and Zolotarev (1873).

Let $b_1, \ldots, b_m$ be an ordered basis of the lattice $L \subset \mathbb{R}^n, m \ge \beta$. We call the basis $b_1, \ldots, b_m$ an β–BKZ basis, i.e. a *block Korkin–Zolotarev basis* with block size β, if it satisfies (1) and if

$$\pi_i(b_i), \pi_i(b_{i+1}), \ldots, \pi_i(b_{i+\beta-1}) \text{ are Korkin–Zolotarev bases for } \quad i = 1, \ldots, m - \beta + 1. \tag{3}$$

Obviously every $(\beta + 1)$–BKZ basis is also a β–BKZ basis. The condition (3) means that the vector $\widehat{b}_i = \pi_i(b_i)$ is a shortest non–zero vector in the lattice $\pi_i(L(b_1, \ldots, b_{\min(i+\beta-1,m)}))$ for $i = 1, \ldots, m$. If (1) holds Condition 3 with $\beta = 3$ is equivalent to

$$\|\widehat{b}_i\|^2 \le \|\pi_i(b_{i+1})\|^2 = \|\widehat{b}_{i+1}\|^2 + \mu_{i+1,i}^2 \|\widehat{b}_i\|^2 \ \text{ for } \ i = 1, \ldots, m - 1. \tag{4}$$

We see that a basis $b_1, \ldots, b_m$ is 2 - BKZ iff it has Properties 1 and 4. These bases have been introduced by A.K. Lenstra, H.W. Lenstra, Jr. and L. Lovász (1982). We call a basis $b_1, \ldots, b_m$ L^3*-reduced* with $\delta \in (\frac{1}{4}, 1]$ if it has Property 1 and if

$$\delta \|\widehat{b}_i\|^2 \le \|\widehat{b}_{i+1}\|^2 + \mu_{i+1,i}^2 \|\widehat{b}_i\|^2 \quad \text{for} \quad i = 1, \ldots, m - 1. \tag{5}$$

LENSTRA et alii (1983) have in particular considered $\delta = 3/4$. From their work we have the following

Theorem 1 *Every basis $b_1, \ldots, b_m$ of lattice L that is L^3–reduced with $\delta \in (\frac{1}{4}, 1]$ satisfies with $\alpha = 1/(\delta - \frac{1}{4})$:*

$$\alpha^{1-i} \leq \|\widehat{b}_i\|^2 \lambda_i(L)^{-2} \leq \|b_i\|^2 \lambda_i(L)^{-2} \leq \alpha^{m-1} \quad \textit{for} \quad i = 1, \ldots, m.$$

In the limit case $\delta = 1$ we have $\alpha = 4/3$, and thus every 2 - BKZ basis $b_1, \ldots, b_m$ of lattice L satisfies for $i = 1, \ldots, m$

$$\left(\frac{3}{4}\right)^{i-1} \leq \|\widehat{b}_i\|^2 \lambda_i(L)^{-2} \leq \|b_i\|^2 \lambda_i(L)^{-2} \leq \left(\frac{4}{3}\right)^{m-1}. \tag{6}$$

A β–BKZ basis $b_1, \ldots, b_m$ with maximal block size $\beta = m$ is a Korkin–Zolotarev basis. For these bases we have the following bounds on $\|b_i\|^2 \lambda_i^{-2}$, where the upper bound is essentially due to MAHLER (1938) and the lower bound is from LAGARIAS, H.W. LENSTRA, JR., SCHNORR (1990).

Theorem 2 *Every Korkin–Zolotarev basis $b_1, \ldots, b_m$ of lattice L satisfies* $\frac{4}{i+3} \leq \|b_i\|^2 \lambda_i(L)^{-2} \leq \frac{i+3}{4}$ *for* $i = 1, \ldots, m$.

We are going to extend Theorems 1 and 2 to arbitrary β–BKZ bases. For this we use the Hermite constant γ_m which is defined as

$$\gamma_m = \sup\{\lambda_1(L)^2 d(L)^{-2/m} : \text{for lattices } L \text{ of rank } m\}.$$

Its value is known for $m \leq 8$, see CASSELS (1971), Appendix. We have $\gamma_2^2 = \frac{4}{3}$, $\gamma_3^3 = 2$, $\gamma_4^4 = 4$, $\gamma_5^5 = 8$, $\gamma_6^6 = 2^6/3$, $\gamma_7^7 = 2^6$, $\gamma_8^8 = 2^8$. It is known that $\gamma_m \leq \frac{2}{3}m$ for $m \geq 2$ and

$$\frac{m}{2\pi e}(1 + o(1)) \leq \gamma_m \leq \frac{0.872m}{\pi e}(1 + o(1)) \quad \text{as} \quad m \to \infty,$$

where the upper bound is due to KABATYANSKII and LEVENSHTEIN (1978), see CONWAY and SLOANE (1988), and the lower bound follows from the Minkowski–Hlawka theorem, see CASSELS (1971), VI. 2.2. Theorems 3 and 4 below will be proved in Section 2.

Theorem 3 *Every β-BKZ basis $b_1, \ldots, b_m$ of lattice L satisfies*

$$\|\widehat{b}_i\|^2 \lambda_i(L)^{-2} \leq \gamma_\beta^{2\frac{m-i}{\beta-1}} \qquad \textit{for } i = 1, \ldots, m$$

$$\|b_i\|^2 \lambda_i(L)^{-2} \leq \gamma_\beta^{2\frac{m-1}{\beta-1}} \; \frac{i+3}{4} \qquad \textit{for } i = 1, \ldots, m.$$

Theorem 3 improves and extends the inequality $\|b_1\|^2 \leq (6\beta^2)^{\frac{m}{\beta}} \lambda_1(L)^2$ of SCHNORR (1987). For $\beta = 2$, $\gamma_2^2 = \frac{4}{3}$ it yields

$$\|\widehat{b}_i\|^2 \lambda_i(L)^{-2} \leq \left(\frac{4}{3}\right)^{m-i} \qquad \text{for } i = 1, \ldots, m$$

$$\|b_i\|^2\lambda_i(L)^{-2} \leq \left(\frac{4}{3}\right)^{m-1} \frac{i+3}{4} \qquad \text{for } i = 1, \ldots, m$$

which is close to the upper bound of Inequality 6. It is remarkable that if the block size β is a fixed fraction of the rank, i.e. if m/β is fixed, then the upper bounds of Theorem 3 are polynomial in β.

We next consider lower bounds for $\|b_i\|^2\lambda_i(L)^{-2}$, i.e. upper bounds for $\lambda_i(L)^2\|b_i\|^{-2}$.

Theorem 4 *Every β-BKZ basis $b_1, \ldots, b_m$ of lattice L satisfies*

$$\lambda_i(L)^2\|b_i\|^{-2} \leq \gamma_\beta^{2\frac{i-1}{\beta-1}} \frac{i+3}{4} \qquad for \quad i = 1, \ldots, m.$$

For $i \leq \beta$ the bounds of Theorems 3 and 4 can be replaced by the stronger bounds of Theorem 2.

If the block size β is a fixed fraction of the rank m the upper bounds in Theorem 4 are polynomial in β. For block size $\beta = 2$, $\gamma_\beta^2 = \frac{4}{3}$ Theorem 4 yields

$$\left(\frac{4}{3}\right)^{i-1} \frac{i+3}{4} \leq \|\widehat{b}_i\|^2\lambda_i(L)^2 \qquad \text{for} \quad i = 1, \ldots, m$$

which is almost the lower bound from Inequality 6.

The values $\gamma_\beta^{\frac{2}{\beta-1}}$, appearing in Theorems 3 and 4, are known for $\beta = 2, \ldots, 8$:

β	2	3	4	5	6	7	8
$\gamma_\beta^{\frac{2}{\beta-1}}$	$\frac{4}{3}$	$2^{1/3}$	$2^{1/3}$	$2^{\frac{3}{10}}$	$2^{\frac{2}{5}}\,3^{-\frac{1}{15}}$	$2^{2/7}$	$2^{1/4}$
$\approx$	1.333	1.260	1.260	1.231	1.226	1.219	1.189

It is interesting to find the minimal constants $c_{\beta,m}$ for which

$$\|b_1\|^2 \ \lambda_1(L)^{-2} \ \leq \ c_{\beta,m}$$

holds for all β–BKZ bases $b_1, \ldots, b_m$ of rank m. Bachem and Kannan (1984) show that $c_{2,m} = \left(\frac{4}{3}\right)^{m-1}$. We prove in Section 5 that $c_{3,m} = \sqrt{\frac{3}{2}}^{m-3}$ holds for odd $m \geq 3$. Notice that the upper bounds in Theorem 3 are not optimal for $\beta \geq 3$.

3 Proofs for Theorems 3 and 4.

Proposition 5 *For every β–BKZ basis $b_1, \ldots, b_m$ with $m \geq \beta$ we have for $M = \max(\|\widehat{b}_{m-\beta+2}\|, \ldots, \|\widehat{b}_m\|)$ that $\|b_1\| \leq \gamma_\beta^{\frac{m-1}{\beta-1}} M$.*

Proof. We extend the basis $b_1, \ldots, b_m$ by $\beta - 2$ additional vectors to

$$b_{-\beta+3}, \ldots, b_{-1}, b_0, b_1, \ldots, b_m \tag{7}$$

such that the following holds:
1. $\|b_i\| = \|b_1\|$ for $i \leq 0$
2. $\langle b_i, b_j \rangle = 0$ for $i \leq 0$, $i \neq j$ and for $j = -\beta + 3, \ldots, m$.

Basis 7 is a β–BKZ basis of rank at least $2(\beta - 1)$. By definition of the Hermite constant γ_β we have

$$\|\widehat{b}_i\|^\beta \leq \gamma_\beta^{\beta/2} \|\widehat{b}_i\| \|\widehat{b}_{i+1}\| \cdots \|\widehat{b}_{i+\beta-1}\| \quad \text{for} \quad i = -\beta + 3, \ldots, m - \beta + 1.$$

Multiplication of these $m - 1$ inequalities yields

$$\|\widehat{b}_{-\beta+3}\|^\beta \|\widehat{b}_{-\beta+4}\|^\beta \cdots \|\widehat{b}_{m-\beta+1}\|^\beta \leq$$

$$\gamma_\beta^{\beta(m-1)/2} \|\widehat{b}_{-\beta+3}\|^1 \|\widehat{b}_{-\beta+4}\|^2 \cdots \|\widehat{b}_1\|^{\beta-1} \|\widehat{b}_2\|^\beta \cdots \|\widehat{b}_{m-\beta+1}\|^\beta$$

$$\|\widehat{b}_{m-\beta+2}\|^{\beta-1} \cdots \|\widehat{b}_{m-1}\|^2 \|\widehat{b}_m\|^1.$$

This implies

$$\|\widehat{b}_{-\beta+3}\|^{\beta-1} \|\widehat{b}_{-\beta+4}\|^{\beta-2} \cdots \|\widehat{b}_0\|^2 \ \|\widehat{b}_1\|^1 \leq$$

$$\gamma_\beta^{\beta(m-1)/2} \|\widehat{b}_{m-\beta+2}\|^{\beta-1} \cdots \|\widehat{b}_{m-1}\|^2 \|\widehat{b}_m\|^1.$$

We have $\|\widehat{b}_{-\beta+3}\| = \cdots = \|\widehat{b}_1\| = \|b_1\|$, hence

$$\|b_1\|^{\binom{\beta}{2}} \leq \gamma_\beta^{\beta(m-1)/2} M^{\binom{\beta}{2}} \quad \text{and thus} \quad \|b_1\| \leq \gamma_\beta^{\frac{m-1}{\beta-1}} M. \qquad \square$$

Corollary 6 *Every β-BKZ basis $b_1, \ldots, b_m$ of the lattice L satisfies*

$$\|b_1\|^2 \leq \gamma_\beta^{2\frac{m-1}{\beta-1}} \ \lambda_1(L)^2 \ .$$

Proof. If $m \leq \beta$ we have $\|b_1\| = \lambda_1(L)$ and the claim holds. Now let $m > \beta$ and let v be a shortest non–zero lattice vector. W.l.o.g. we can assume that $v \notin L(b_1, \ldots, b_{m-1})$ for otherwise we can reduce m and the claim holds by induction on m. If $v \notin L(b_1, \ldots, b_{m-1})$ we have

$$\lambda_1(L) = \|v\| \geq \lambda_1(L_i) = \|\widehat{b}_i\| \quad \text{for} \quad i = m - \beta + 1, \ldots, m.$$

This shows that $\lambda_1(L) \geq \max(\|\widehat{b}_{m-\beta+2}\|, \ldots, \|\widehat{b}_m\|)$ and thus the claim follows from Proposition 5. $\square$

Proof of Theorem 3. Corollary 6 applied to the lattice $L_i = \pi_i(L)$ yields the inequalities

$$\|\widehat{b}_i\|^2 \leq \gamma_\beta^{2\frac{m-i}{\beta-1}} \lambda_1(L_i)^2 \quad \text{for} \quad i = 1, \ldots, m. \tag{8}$$

Moreover we have $\lambda_1(L_i) \leq \lambda_i(L)$. This is because there are i linearly independent vectors of length at most $\lambda_i(L)$ in L and at least one of these vectors v satisfies $\pi_i(v) \neq 0$. Hence $\lambda_1(L_i) \leq \|\pi_i(v)\| \leq \lambda_i(L)$. Using Inequality 8 and $\mu_{i,j}^2 \leq 1/4$ we obtain for $i = 1, \ldots, m$

$$\begin{aligned}
\|b_i\|^2 &= \|\widehat{b}_i\|^2 + \sum_{j=1}^{i-1} \mu_{i,j}^2 \, \|\widehat{b}_j\|^2 \\
&\leq \gamma_\beta^{2\frac{m-i}{\beta-1}} \lambda_i(L)^2 + \frac{1}{4} \sum_{j=1}^{i-1} \gamma_\beta^{2\frac{m-j}{\beta-1}} \lambda_j(L)^2 \\
&\leq \gamma_\beta^{\frac{2m}{\beta-1}} \left(\gamma_\beta^{2\frac{-i}{\beta-1}} + \frac{1}{4} \sum_{j=1}^{i-1} \gamma_\beta^{2\frac{-j}{\beta-1}} \right) \lambda_i(L)^2 \\
&\leq \gamma_\beta^{2\frac{m-1}{\beta-1}} \frac{i+3}{4} \lambda_i(L)^2 \ .
\end{aligned}$$

This proves the second inequalities of Theorem 3. The first inequalities in Theorem 3 follows from 8 and $\lambda_1(L_i) \leq \lambda_i(L)$.
Note that the factor $\frac{i+3}{4}$ in Theorem 3 can be replaced by the smaller factor $\frac{\gamma}{4\gamma-4}$ with $\gamma = \gamma_\beta^{\frac{2}{\beta-1}}$. This follows from an explicit evaluation of the last sum in the proof of Theorem 3. □

Proof of Theorem 4. By definition of $\lambda_i = \lambda_i(L)$ we have

$$\lambda_i^2 \leq \max\{\|b_j\|^2 \text{ for } j = 1, \ldots, i\} \ .$$

It follows from $\|b_j\|^2 \leq \|\widehat{b}_j\|^2 + \sum_{k=1}^{j-1} \mu_{j,k}^2 \|\widehat{b}_k\|^2$ and $\mu_{i,j}^2 \leq 1/4$ that

$$\lambda_i^2 \leq \frac{i+3}{4} \max\{\|\widehat{b}_j\|^2 \text{ for } j = 1, \ldots, i\} \ . \tag{9}$$

Proposition 5 applied to the β–BKZ basis $\pi_j(b_j), \ldots, \pi_j(b_i)$ yields the inequalities

$$\|\widehat{b}_j\| \leq \gamma_\beta^{\frac{i-j}{\beta-1}} \max\{\|\widehat{b}_{i-\beta+2}\|, \ldots, \|\widehat{b}_i\|\} \quad \text{for} \quad 1 \leq j \leq i - \beta + 1 \tag{10}$$

On the other hand we have for $i - \beta + 2 \leq j \leq i$

$$\|\widehat{b}_j\| \leq \|\pi_j(b_i)\| \leq \|b_i\| \ .$$

From this and in Equality 10 we see that

$$\|\widehat{b}_j\| \leq \gamma_\beta^{\frac{i-j}{\beta-1}} \|b_i\| \qquad \text{for} \qquad 1 \leq j \leq i \ .$$

These inequalities and Inequality 9 yield Theorem 4:

$$\lambda_i^2 \leq \frac{i+3}{4} \gamma_\beta^{2\frac{i-1}{\beta-1}} \|b_i\|^2 \qquad \text{for} \quad i = 1, \ldots, m \ . \qquad \square$$

4 Finding nearby lattice points

BABAI (1986) shows how Lovász basis reduction can be used to find a point of a given lattice L that is nearest within a factor $2^{m/2}$ to a given point in span(L). BABAI poses the question whether the factor $2^{m/2}$ can be improved using block Korkin–Zolotarev bases. We answer this question in the affirmative.

We let α_β denote the maximum of $\|b_1\|^2/\|\hat{b}_\beta\|^2$ over all Korkin–Zolotarev bases $b_1, \ldots, b_\beta$. The inequality $\alpha_\beta \le \beta^{1+\ln \beta}$ has been shown in SCHNORR (1987).

Theorem 7 *Let $b_1, \ldots, b_m \in \mathbb{Z}^n$ be a β–BKZ basis of lattice L and let $x = \sum_{i=1}^m x_i b_i$ be in* span$(b_1, \ldots, b_m)$. *If the lattice point $v = \sum_{j=1}^m v_j b_j$ satisfies $|x_j - \sum_{i=j}^m v_i \mu_{i,j}| \le 1/2$ for $j = 1, \ldots, m$, then we have*

$$\|x - v\|^2 \le C_{m,\beta} \min_{u \in L} \|x - u\|^2 \ , \ \textit{where } C_{m,\beta} = m \ \gamma_\beta^{2\frac{m-1}{\beta-1}} \ \alpha_{\beta-1} \ .$$

Proof. The definition of v implies that

$$\|x - v\|^2 \ \le \ (\|\hat{b}_1\|^2 + \cdots + \|\hat{b}_m\|^2)/4 \ .$$

From Proposition 5 we have the inequality

$$\|\hat{b}_i\|^2 \ \le \ \gamma_\beta^{2\frac{m-i}{\beta-1}} \ \max(\|\hat{b}_{m-\beta+2}\|^2, \ldots, \|\hat{b}_m\|^2) \text{ for } 1 \le i \le m - \beta + 1 \ .$$

This yields

$$\begin{aligned} \|x - v\|^2 \ &\le \ \frac{1}{4} \sum_{j=1}^m \gamma_\beta^{2\frac{j-1}{\beta-1}} \ \max(\|\hat{b}_{m-\beta+2}\|^2, \ldots, \|\hat{b}_m\|^2) \\ &\le \ \frac{1}{4} \sum_{j=1}^m \gamma_\beta^{2\frac{j-1}{\beta-1}} \ \alpha_{\beta-1} \ \|\hat{b}_m\|^2 \\ &\le \ \frac{m}{4} \ \gamma_\beta^{2\frac{m-1}{\beta-1}} \ \alpha_{\beta-1} \ \|\hat{b}_m\|^2 \ . \end{aligned} \tag{11}$$

Let $u \in L$ be the lattice point that is nearest to x. Following BABAI we consider two cases.

CASE a: $u - v \in L(b_1, \ldots, b_{m-1})$. Then $u - v$ is a nearest lattice point to $x' - v$ in $L(b_1, \ldots, b_{m-1})$ where $x' \in$ span$(b_1, \ldots, b_{m-1})$ is the orthogonal projection of x, $x' = x - \pi_{m-1}(x)$. Induction on m yields

$$\begin{aligned} \|x - v\|^2 \ &= \ \|x - x'\|^2 + \|x' - v\|^2 \\ &\le \ (1 + C_{m-1,\beta}) \ \|x - u\|^2 \\ &\le \ C_{m,\beta} \ \|x - u\|^2 \end{aligned}$$

CASE b: $u - v \notin L(b_1, \ldots, b_{m-1})$. In this case we have

$$\|\hat{b}_m\|^2/4 \le \|x - u\|^2 \ .$$

Hence we have from Inequality 11 that

$$\begin{aligned} \|x-v\|^2 &\leq \frac{m}{4}\, \gamma_\beta^{2\frac{m-1}{\beta-1}}\, \alpha_{\beta-1}\, 4\, \|x-u\|^2 \\ &= C_{m,\beta}\, \|x-u\|^2 \ . \end{aligned} \qquad \square$$

We finally reduce with a different construction for v the constant $C_{m,\beta}$ of Theorem 7 to $m\, \gamma_\beta^{2\frac{m-1}{\beta-1}}$. The improved constant $m\, \gamma_\beta^{2\frac{m-1}{\beta-1}}$ is polynomial in β if β is a fixed fraction of m.

Theorem 8 *Let* $b_1, \ldots, b_m \in \mathbb{Z}^n$ *be a* β*–BKZ basis and* $x = \sum_{i=1}^m x_i b_i$ *. Suppose that* $\|\widehat{b}_k\| = \max(\|\widehat{b}_{m-\beta+1}\|, \ldots, \|\widehat{b}_m\|)$, $m-\beta+1 \leq k \leq m$ *. Let* $v = \sum_{i=1}^m v_i b_i$ *be a lattice point such that* $\sum_{j=k}^m |x_j - \sum_{i=j}^m v_i \mu_{i,j}|^2\, \|\hat{b}_j\|^2$ *is minimal for all* $v_k, \ldots, v_m \in \mathbb{Z}$ *and let* $|x_j - \sum_{i=j}^m v_i \mu_{i,j}| \leq \frac{1}{2}$ *for* $j = k-1, \ldots, 1$ *. Then we have* $\|x-v\|^2 \leq m\, \gamma_\beta^{2\frac{m-1}{\beta-1}}\, \min_{u \in L} \|x-u\|^2$.

Proof. Let $u \in L$ be the lattice point that is nearest to x.
CASE a: $u - v \in L(b_1, \ldots, b_{m-1})$. Then $u - v$ is a nearest lattice point to $x' - v$ in $L(b_1, \ldots, b_{m-1})$ where $x' \in \mathrm{span}(b_1, \ldots, b_{m-1})$ is the orthogonal projection of x, $x' = x - \pi_{m-1}(x)$. Induction on m yields

$$\begin{aligned} \|x-v\|^2 &= \|x-x'\|^2 + \|x'-v\|^2 \\ &\leq (1 + (m-1)\gamma_\beta^{2\frac{m-2}{\beta-1}})\, \|x-u\|^2 \\ &\leq m\, \gamma_\beta^{2\frac{m-1}{\beta-1}}\, \|x-u\|^2 \ . \end{aligned}$$

CASE b: $u - v \notin L(b_1, \ldots, b_{m-1})$. It follows from the choice of v and k that

$$\|x-u\|^2 \geq \frac{1}{4}\, \lambda_1(L_k)^2 = \frac{1}{4}\|\widehat{b}_k\|^2 \geq \frac{1}{4}\, \max(\, \|\hat{b}_{m-\beta+2}\|^2, \ldots, \|\hat{b}_m\|^2\,) \ .$$

As in the proof of Theorem 7 we have

$$\begin{aligned} \|x-v\|^2 &\leq (\, \|\hat{b}_1\|^2 + \cdots + \|\hat{b}_m\|^2\,)\, /\, 4 \\ &\leq \frac{1}{4} \sum_{j=1}^m \gamma_\beta^{2\frac{j-1}{\beta-1}}\, \max(\|\hat{b}_{m-\beta+2}\|^2, \ldots, \|\hat{b}_m\|^2) \ . \end{aligned}$$

Consequently

$$\begin{aligned} \|x-v\|^2 &\leq \sum_{j=1}^m \gamma_\beta^{2\frac{j-1}{\beta-1}}\, \|x-u\|^2 \\ &\leq m\, \gamma_\beta^{2\frac{m-1}{\beta-1}}\, \|x-u\|^2 \ . \end{aligned} \qquad \square$$

Computational bounds. Computing a lattice vector $v = \sum_{i=1}^{m} v_i b_i$ as in Theorem 8 can be done using the NEAREST VECTOR ALGORITHM of KANNAN (1987). Given a β–BKZ basis $b_1, \ldots, b_m$ this algorithm enumerates $\beta^{O(\beta)}$ many lattice vectors close to x and finds integers $v_{m-\beta+1}, \ldots, v_m$ that minimize $\sum_{j=m-\beta+1}^{m} |x_j - \sum_{i=j}^{m} v_i \mu_{i,j}|^2 \, \|\hat{b}_i\|^2$. An "approximate" β–BKZ basis can be computed from an arbitrary lattice basis $b_1, \ldots, b_m \in \mathbb{Z}^n$ by the algorithm of SCHNORR (1987). This algorithm finds an "approximate" β–BKZ basis using $O(nm(\beta^{O(\beta)} + m^2)\log B)$ arithmetic operations on $O(m \log B)$ bit integers where B is the maximal length of the given basis vectors. This algorithm is theoretical.

For practical algorithms performing block Korkin–Zolotarev reduction see SCHNORR, EUCHNER (1991). The Schnorr, Euchner algorithm transforms an arbitrary lattice basis into a δ–*approximate* β–BKZ basis, for some fixed $\delta < 1$, which by definition satisfies for $i = 1, \ldots, m$

$$\delta \, \|\widehat{b}_i\|^2 \leq \lambda_1(\pi_i L(b_1, \ldots, b_{\min(i+\beta-1,m)}))^2 \ .$$

The Schnorr, Euchner algorithm is not proven to be polynomial time, but in practice its time bound appears to be $O(nm(\beta^{O(\beta)} + m^2)\log B)$. No polynomial time algorithm is known for $\delta = 1$, not even in the case that β is 2.

5 Critical β–BKZ bases for block size 2 and 3

We call a β–BKZ basis $b_1, \ldots, b_m$ of the lattice L *critical* for β and m if the ratio $\|b_1\| \,/\, \lambda_1(L)$ is maximal for all β–BKZ bases of rank m. BACHEM and KANNAN (1984) present critical 2–BKZ bases $b_1, \ldots, b_m$. We establish critical 3–BKZ bases $b_1, \ldots, b_m$. We evaluate the constant $\alpha_3 = \max \|b_1\|^2 \,/\, \|\widehat{b}_3\|^2$, where the maximum is taken over all Korkin–Zolotarev bases b_1, b_2, b_3. We prove $\alpha_3 = \frac{3}{2}$.

We describe a slight variant of the critical 2–BKZ bases of BACHEM, KANNAN (1984). Let $\rho = \sqrt{\frac{3}{4}}$. We define the basis matrix $A_m = [b_1, \ldots, b_m] \in M_{m,m}(\mathbb{R})$ as

$$A_m = \begin{bmatrix} 1 & \frac{1}{2} & & & & & \\ & \rho & \rho/2 & & & \mathrm{O} & \\ & & \rho^2 & \rho^2/2 & & & \\ & & & \ddots & \ddots & & \\ & & & & \ddots & \ddots & \\ & & \mathrm{O} & & & \rho^{m-2} & \rho^{m-2}/2 \\ & & & & & & \rho^{m-1} \end{bmatrix} \tag{12}$$

The matrices A_m satisfy the recursion

$$A_m = \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ & \vdots & \cdots\cdots \\ 0 & \vdots & \rho \cdot A_{m-1} \\ & \vdots & \end{bmatrix} .$$

Theorem 9 (Bachem, Kannan *1984) The column vectors of the matrix A_m form a critical 2–BKZ basis.*

Proof. It can easily be seen that the vector b_m is a shortest non–zero vector in $L(b_1, \ldots, b_m) - L(b_1, \ldots, b_{m-1})$. It follows that b_m is the shortest non–zero vector in the lattice $L(b_1, \ldots, b_m)$. We have

$$\lambda_1^2 = \|b_m\|^2 = \rho^{2m-4}(\rho^2 + \frac{1}{4}) = (\frac{3}{4})^{m-2} .$$

Therefore $\lambda_1 = \lambda_1(L(b_1, \ldots, b_m))$ satisfies

$$\|b_1\|^2 \lambda_1^{-2} = \left(\frac{4}{3}\right)^{m-2} .$$

Now let $b_1, \ldots, b_m$ be an arbitrary 2–BKZ basis. Let $v = \sum_{i=1}^m t_i b_i$ be a shortest non–zero lattice vector and $\mu = \max\{i \mid t_i \neq 0\}$. Then $\lambda_1 = \lambda_1(L(b_1, \ldots, b_m))$ satisfies

$$\lambda_1^2 \overset{1.}{\geq} \|\pi_{\mu-1}(v)\|^2 \overset{2.}{\geq} \|\widehat{b}_{\mu-1}\|^2 \overset{3.}{\geq} \left(\frac{3}{4}\right)^{\mu-2} \|b_1\|^2 .$$

1. holds since $\pi_{\mu-1}(v) \neq 0$, 2. holds since $\pi_{\mu-1}(b_{\mu-1})$ is minimal in the lattice $\pi_{\mu-1} L(b_{\mu-1}, b_\mu)$ which contains $\pi_{\mu-1}(v)$, 3. follows from the inequality 6. We now see that

$$\|b_1\|^2 \lambda_1^{-2} \leq \left(\frac{4}{3}\right)^{\mu-2} \leq \left(\frac{4}{3}\right)^{m-2} .$$

This shows that the columns of Matrix 12 form a critical basis. □

A sequence of 3–BKZ bases. The column vectors b_1, b_2 of the basis matrix A_2 form a lattice packing of maximal density for rank 2, i.e. $L^{(2)} = L(b_1, b_2)$ satisfies $\lambda_1(L^{(2)})^2(\det L^{(2)})^{-1} = \gamma_2 = \sqrt{\frac{3}{2}}$. The matrices A_m extend A_2 periodically with period 1 to arbitrary rank $m \geq 2$. The following construction of 3–BKZ bases starts with a basis matrix $B_3 = [b_1, b_2, b_3]$ of the lattice $L^{(3)} = L(b_1, b_2, b_3)$ that forms a lattice packing of maximal density for dimension 3, i.e. $\lambda_1(L^{(3)})^2(\det L^{(3)})^{-\frac{2}{4}} = \gamma_3 = \sqrt{2}$. We extend B_3 periodically with period 2 to arbitrary dimension $m \geq 3$.

For arbitrary $m = 1, 2, \ldots$ we define the $m \times m$ matrices

$$B_m = \mathrm{diag}(d_1, \ldots, d_m) M_m \text{ with } M_m = [\mu_{i,j}]_{1 \leq i,j \leq m}^\top$$

where $\operatorname{diag}(d_1, \ldots, d_m)$ is the $m \times m$ diagonal matrix with positive diagonal entries $d_1, \ldots, d_m$ and M_m is an upper triangular matrix with ones on the diagonal. It follows that the $\mu_{i,j}$ are the Gram–Schmidt coefficients of the basis $b_1, \ldots, b_m$ consisting of the column vectors of B_m. The matrices M_m, B_m are upper triangular matrices with three non–zero diagonals. They satisfy for $m \geq 4$ the following recursion

$$M_m = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & \ldots \\ & 1 & -\frac{1}{3} & \frac{1}{3} & 0 & \ldots \\ & & \ldots\ldots\ldots\ldots & & & \\ & & \vdots & & & \\ \mathrm{O} & & \vdots & & M_{m-2} & \\ & & \vdots & & & \end{bmatrix}$$

$$B_m = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & \ldots \\ & \frac{\sqrt{3}}{2} & \frac{-1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & 0 & \ldots \\ & & \ldots\ldots\ldots\ldots\ldots\ldots & & & \\ & & \vdots & & & \\ & & \vdots & & & \\ \mathrm{O} & & \vdots & & \sqrt{\frac{2}{3}}\, B_{m-2} & \\ & & \vdots & & & \\ & & \vdots & & & \end{bmatrix}$$

Also the matrix M_m (B_m, resp.) has the submatrix M_{m-2} (B_{m-2}, res p.) in its upper left corner. For example, we have

$$\begin{aligned} B_4 &= \begin{bmatrix} 1 & & & \\ & \frac{\sqrt{3}}{2} & & \mathrm{O} \\ \mathrm{O} & & \sqrt{\frac{2}{3}} & \\ & & & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ & 1 & -\frac{1}{3} & \frac{1}{3} \\ \mathrm{O} & & 1 & \frac{1}{2} \\ & & & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ & \frac{\sqrt{3}}{2} & \frac{-1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} \\ & \mathrm{O} & \sqrt{\frac{2}{3}} & \frac{1}{2}\sqrt{\frac{2}{3}} \\ & & & \frac{1}{\sqrt{2}} \end{bmatrix} \end{aligned} \tag{13}$$

The diagonal coefficients d_j and the Gram-Schmidt coefficients satisfy for $i = 1, 2, \ldots$

$$d_{2i-1} = \sqrt{\frac{2}{3}}^{i-1} \quad , \quad d_{2i} = \frac{\sqrt{3}}{2}\, d_{2i-1}$$

$$\mu_{2i,2i-1} = \mu_{2i+1,2i-1} = \frac{1}{2} \ , \ \mu_{2i+k,2i-1} = 0 \text{ for } k \geq 2$$

$$-\mu_{2i+1,2i} = \mu_{2i+2,2i} = \frac{1}{3} \ , \ \mu_{2i+k,2i} = 0 \quad \text{for} \quad k \geq 3 \ .$$

Let $b_1, \ldots, b_m$ be the column vectors of B_m, $B_m = [b_1, \ldots, b_m]$ and let $L^{(m)} = L(b_1, \ldots, b_m)$ denote the lattice with basis $b_1, \ldots, b_m$.

Theorem 10 *The column vectors $b_1, \ldots, b_m$ of B_m form a 3–BKZ basis.*

Proof. The vectors b_1, b_2, b_3 and the vectors $\pi_2(b_2), \pi_2(b_3), \pi_2(b_4)$ form Korkin–Zolotarev bases. This can easily be verified. We see from the recursive structure of B_m that the triple of vectors $(\pi_i(b_i), \pi_i(b_{i+1}), \pi_i(b_{i+2}))$ is isometric to the triple

$$(b_1, b_2, b_3)\sqrt{\frac{2}{3}}^{(i-1)/2} \qquad \text{for odd } i \ ,$$

$$(\pi_2(b_2), \pi_2(b_3), \pi_2(b_4))\sqrt{\frac{2}{3}}^{(i-2)/2} \qquad \text{for even } i \ .$$

Hence $\pi_i(b_i)$, $\pi_i(b_{i+1})$, $\pi_i(b_{i+2})$ is a Korkin–Zolotarev basis for $i = 1, \ldots, m-2$. □

Lemma 11 *The lattices $L^{(m)}$ satisfy $\lambda_1(L^{(m)}) = \|b_m\|$ for $m = 1, 2, \ldots$, where $\|b_m\|^2 = \sqrt{\frac{3}{2}}^{m-3}$ for odd $m \geq 3$ and $\|b_m\|^2 = \frac{4}{3}\sqrt{\frac{3}{2}}^{m-3}$ for even $m \geq 4$.*

Proof. It is sufficient to show that b_m is a shortest non–zero vector in the lattice $L^{(m)} = L(b_1, \ldots, b_m)$. This claim follows from

$$\lambda_1(L^{(m)}) \geq \lambda_1(\pi_{m-2}(L^{(m)})) = \|\widehat{b}_{m-2}\| = \|b_m\| \ .$$

The inequality holds because of the periodic structure of the basis matrix B_m. The first equality holds since $b_1, \ldots, b_m$ is a 3–BKZ basis. □

In order to show that the basis $b_1, \ldots, b_m$ of lattice $L^{(m)}$ is a critical 3–BKZ basis we need an upper bound on $\|b_1\| \,/\, \lambda_1(L)$ which holds for all 3–BKZ bases of all lattices L. Let α_β be the lattice constant

$$\alpha_\beta = \max \|b_1\|^2 \ \|\widehat{b}_\beta\|^{-2}$$

where the maximum is taken over all Korkin–Zolotarev bases $b_1, \ldots, b_\beta$. The following theorem improves the upper bound $\|b_1\|^2 \lambda_1(L)^{-2} \leq \alpha_\beta^k$ from Schnorr (1987).

Theorem 12 *Every β–BKZ basis $b_1, \ldots, b_m$ of lattice L, where $k = (m-1)/(\beta-1)$ is integer, satisfies $\|b_1\|^2 \lambda_1(L)^{-2} \leq \alpha_\beta^{k-1}$.*

Proof. Let $v = \sum_{i=1}^{m} t_i b_i$ be a shortest non–zero vector in L. We can assume that $v \notin L(b_1, \ldots, b_{m-\beta+1})$ since otherwise we can reduce k and the induction hypothesis for $k-1$ yields $\|b_1\|^2 \lambda_1(L)^{-2} \leq \alpha_\beta^{k-2}$. From $v \notin L(b_1, \ldots, b_{m-\beta+1})$ we see that $\pi_{m-\beta+1}(v) \neq 0$. Since
$\pi_{m-\beta+1}(b_{m-\beta+1}), \ldots, \pi_{m-\beta+1}(b_m)$ is a Korkin–Zolotarev basis we have

$$\|\widehat{b}_{m-\beta+1}\| \leq \|\pi_{m-\beta+1}(v)\| \leq \lambda_1(L) \ .$$

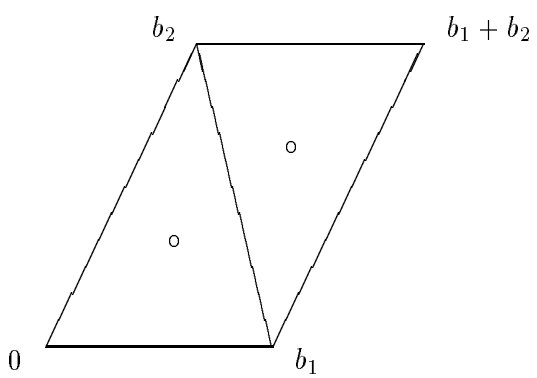
Moreover $\|\widehat{b}_{1+(\beta-1)(j-1)}\|^2 \leq \alpha_\beta \, \|\widehat{b}_{1+(\beta-1)j}\|^2$ holds for $j = 1, \ldots, k-1$ since the basis $b_1, \ldots, b_m$ is β–BKZ. Therefore

$$\|b_1\|^2 \leq \alpha_\beta^{k-1} \, \|\widehat{b}_{m-\beta+1}\|^2 \leq \alpha_\beta^{k-1} \, \lambda_1(L)^2 \ . \qquad \square$$

Theorem 13 $\alpha_3 = \frac{3}{2}$, *i.e.* $\alpha_3^{-1/2}$ *is the height of the tetrahedron with side length* 1.

Proof. Let b_1, b_2, b_3 be a Korkin–Zolotarev basis with $\|b_1\| = 1$ and minimal $\|\widehat{b}_3\|$, i.e. $\alpha_3 = \|b_1\|^2 \, \|\widehat{b}_3\|^{-2} = \|\widehat{b}_3\|^{-2}$. Consider the orthogonal projection $\mu_{3,1} b_1 + \mu_{3,2} \widehat{b}_2$ of b_3 in $\operatorname{span}(b_1, b_2)$. We claim that $\mu_{3,1} b_1 + \mu_{3,2} \widehat{b}_2$ must be a *deep hole* for the lattice $L(b_1, b_2)$, i.e. a point that has maximal distance from the lattice points. If $\mu_{3,1} b_1 + \mu_{3,2} \widehat{b}_2$ is not a deep hole for the lattice $L(b_1, b_2)$ we can change $\mu_{3,1}, \mu_{3,2}$ such that the minimal distance of $\mu_{3,1} b_1 + \mu_{3,2} \widehat{b}_2$ from $L(b_1, b_2)$ increases. This permits to decrease $\|\widehat{b}_3\|$ without violating the properties of Korkin–Zolotarev bases.

There are at most two deep holes in the ground mesh of the basis b_1, b_2, namely the points that have equal distance to $0, b_1, b_2$ ($b_1, b_2, b_1 + b_2$, respectively).



deep holes ∘ in the ground mesh

W.l.o.g. we can assume that $\mu_{3,1}b_1 + \mu_{3,2}\widehat{b}_2$ has equal distance from $0, b_1$ and b_2. It follows from $\|b_2\|^2 = \mu_{2,1}^2 + \|\widehat{b}_2\|^2 \geq \|b_1\|^2 = 1$, $|\mu_{2,1}| \leq \frac{1}{2}$ that

$$\|\widehat{b}_2\| \geq \frac{\sqrt{3}}{2} \ . \tag{14}$$

Let $\|\widehat{b}_2\| = \frac{\sqrt{3}}{2} + \varepsilon$ with $\varepsilon \geq 0$. We have

$$\begin{aligned} \|\mu_{3,2}\widehat{b}_2\| &= \frac{\sqrt{3}}{6} \quad \text{if} \quad \varepsilon = 0 \\ \|\mu_{3,2}\widehat{b}_2\| &\leq \frac{\sqrt{3}}{6} + \varepsilon \quad \text{for} \quad \varepsilon \geq 0 \ . \end{aligned} \tag{15}$$

This is because $\varepsilon = 0$ implies that $\|b_1\| = \|b_2\| = \|b_1 - b_2\| = 1$ and thus $\mu_{3,1}b_1 + \mu_{3,2}\widehat{b}_2 = \frac{b_1+b_2}{3}$. Moreover the $\widehat{b}_2$–coordinate of the deep hole $\mu_{3,1}b_1 + \mu_{3,2}\widehat{b}_2$ does not increase faster in ε than $\|\widehat{b}_2\|$. We see from $\|\pi_2(b_3)\| \geq \|\widehat{b}_2\|$ that $\|\widehat{b}_3\|^2 + \mu_{3,2}^2\|\widehat{b}_2\|^2 \geq \|\widehat{b}_2\|^2$, and thus

$$\begin{aligned} \|\widehat{b}_3\|^2 &\geq \|\widehat{b}_2\|^2 - \mu_{3,2}\|\widehat{b}_2\|^2 \\ &\overset{15,16}{\geq} \left(\frac{\sqrt{3}}{2} + \varepsilon\right)^2 - \left(\frac{\sqrt{3}}{6} + \varepsilon\right)^2 \\ &\geq \frac{3}{4} - \frac{1}{12} = \frac{2}{3} \end{aligned}$$

holds for the basis matrix $[b_1, b_2, b_3]$. This proves that $\alpha_3 = \frac{3}{2}$. □

Theorem 14 *The lattice basis $b_1, \ldots, b_{2k+1}$ defined by the basis matrix B_{2k+1} is a critical 3–BKZ basis for $k = 1, 2, \ldots$.*

Proof. Lemma 11 shows that $\|b_1\| \,/\, \lambda_1(L^{(2k+1)}) = \sqrt{\frac{3}{2}}^{k-1}$. On the other hand every 3–BKZ basis $b'_1, \ldots, b'_{2k+1}$ of lattice L satisfies $\|b'_1\| \,/\, \lambda_1(L) \leq \sqrt{\alpha_3}^{k-1}$ by Theorem 12, where $\alpha_3 = \frac{3}{2}$ by Theorem 13. This shows that $\|b_1\| \,/\, \lambda_1(L^{(2k+1)})$ is maximal for 3–BKZ bases of rank $2k+1$. □

We conjecture that the lattice bases $B_m = [b_1, \ldots, b_m]$ are critical 3–BKZ bases for even m, too. Theorem 14 only solves the case of odd m.

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