A Framework for Cumulative Default Logics

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TR-92-042

July 1992

Abstract

We present a framework for default reasoning which has its roots in Reiter's Default Logic. Contrary to Reiter, however, we do not consider defaults as inference rules used to generate extensions of a classical set of facts. In our approach defaults are elements of the logical language, and we will define inference rules on defaults. This has several advantages. First of all, we can reason about defaults, not just with defaults. This makes it easy to include different intuitions about the right behaviour of a default logic in an explicit form. Secondly, we can show how some of the problems of Reiter's logic and of some recent proposals to solve them can be handled adequately by exploiting the dependency information contained in derived defaults.

1 Background and Motivation

Reiter's Default Logic DL [14] is currently one of the most popular and most widely used formalizations of default reasoning. There are at least two reasons for this popularity. Firstly, although the technical definition of DL, in particular the fixed point definition of extensions, is rather tricky, the idea underlying the representation of defaults is simple, intuitive and attractive: represent defaults as inference rules with an additional consistency check. Secondly, DL is very expressive. It can, for instance, be used to formalize logic programs with negation [1], a task for which some of its competitor's, e.g. circumscription, turn out to be insufficient. This expressiveness is mainly due to the representation of defaults as inference rules which makes it possible to avoid problems with contraposition of defaults. For instance, from

Computer scientists typically do not know much about default logic.

we probably do not want to conclude the contrapositive default

Who knows much about default logic typically is not a computer scientist.

Such a derivation can be avoided in default logic.

DL assumes knowledge to be given in the form of a default theory (D, W), where W, the facts, is a set of first order formulas, and D, the defaults, a set of structures of the form

$$p{:}\, q_1, \ldots, q_n / r$$

Here p, q_i and r are closed first order formulas. The intuitive reading of the default is: if p is provable and q_i is consistent with what is provable $(1 \leq i \leq n)$ then derive r. p is called prerequisite, q_i justification or consistency condition, and r consequent of the default. We will sometimes use a default d with free variables to represent all closed instances of d.

For a given default theory T default logic produces a set of extensions representing acceptable sets of conclusions a reasoner might adopt. Intuitively, a set of formulas E qualifies as an extension if

1. it contains the facts W,
2. it is deductively closed,
3. it contains the conclusions of all "applicable" default rules, and
4. it contains no "ungrounded" formula that cannot be derived from W and conclusions of applicable defaults in a noncircular way.

To capture this intuition adequately Reiter defines extensions as fixed points of an operator Γ as follows:

Definition 1 *(DL extension)*
Let (D,W) be a default theory, S a set of formulas. Let $\Gamma(S)$ be the smallest set such that:

1. $W \subseteq \Gamma(S)$,
2. $Th(\Gamma(S)) = \Gamma(S)$,
3. *if* $p{:}q_1,...,q_n/r \in D$, $p \in \Gamma(S)$, $\neg q_i \notin S$ $(1 \leq i \leq n)$, *then* $r \in \Gamma(S)$.

E is an extension of (D,W) iff E is a fixed point of Γ.

Based on the notion of extensions a skeptical inference relation for DL can be introduced where a formula is considered derivable from a default theory T iff it is contained in all extensions of T.

Unfortunately, Reiter's logic also has its drawbacks, as has been discussed in various papers, e.g. [13, 9, 2, 5]:

1. Existence of extensions is not guaranteed.
2. The consistency conditions (justifications) of defaults applied within one Reiter extension are not jointly consistent with the generated extension. Reiter's fixed point definition only guarantees that each consistency condition in isolation is consistent with the extension. This leads to counterintuitive conclusions as discussed in [13].
3. The use of inference rules as defaults makes it impossible to reason by cases in DL. Consider the following example
   (a) $Italian{:}Likes\_Wine/Likes\_Wine$
   (b) $French{:}Likes\_Wine/Likes\_Wine$
   (c) $Italian \vee French$

   Intuitively one would expect to derive $Likes\_Wine$ since no matter whether the person at hand is Italian or French one of the defaults should apply. In DL, however, a default can only be applied if its prerequisite has already been derived.
4. DL's skeptical inference relation is not cumulative [9], that is the addition of formulas contained in all extensions of a default theory (D,W) to W may change the set of generated extensions and hence the skeptically derivable formulas. This makes it questionable whether the skeptical derivability relation of DL can be characterized as an inference relation at all. What we usually expect from an inference relation is that it makes knowledge which is implicit in the premises explicit, but adding implicit knowledge in an explicit form to the premises should not change the inferences. Note that cumulativity is an essential property if we want to be able to use derived formulas as Lemmata.

In an earlier paper [2] we have particularly addressed problems 2) and 4) of the above enumeration. The solution of these problems, the logic CDL, was based on a shift from simple propositions to more complicated structures called *assertions*. An assertion is a pair (p, X) consisting of a first-order formula p and a (finite) set of formulas X, the support of p. Basically, the support is used to keep track of the consistency conditions needed to derive a conclusion by default. The intuitive meaning of the assertion is: p is believed since X is consistent with what is believed and the consistency conditions of other believed fomulas. Given an assertional default theory (D, W), where D is a set of Reiter defaults and W a set of assertions, extensions can be defined using a Reiter-like fixed point construction as follows [2]:

Definition 2 *(CDL extension)*
An extension of an assertional default theory (D, W) is a fixed point of the operator Γ which, given a set of assertions S, produces the smallest set of assertions S' such that

1. $W \subseteq S'$,
2. $Th_S(S') = S'$,
3. *if* $p{:}q/r \in D, (p, \{j_1, \ldots, j_k\}) \in S'$,
   and $\{q, r\} \cup Cons(S) \cup Just(S)$ *is consistent,*
   then $(r, \{j_1, \ldots, j_k, q, r\}) \in S'$.

Here

$$Cons(S) = \{p \mid (p, X) \in S\}$$

and

$$Just(S) = \{q \mid (p, Q) \in S, q \in Q\}$$

$Th_S(A)$ is the smallest set such that $A \subseteq Th_S(A)$ and if $(p_1, J_1), \ldots, (p_k, J_k) \in Th_S(A)$ and $p_1, \ldots, p_k \vdash q$, then $(q, J_1 \cup \ldots \cup J_k) \in Th_S(A)$. [1]

In [2] CDL is shown to satisfy existence of extensions, semi-monotony, and cumulativity. A further advantage of the logic is that it allows for a syntactic distinction between hard facts and weak conclusions. An Etherington-style semantics for CDL has been presented in [16]. The ideas underlying CDL have been applied in [4] to define a cumulative inference relation for TMS and logic programs.

Unfortunately CDL's solution to problems 2) and 4) introduces a new problem [4]: the floating conclusions problem. Floating conclusions are conclusions that appear in every extension, but with different supports. Consider the following assertional default theory:

[1] $\vdash$ here and in the rest of the paper stands for classical provability.

$:\neg p/q$
$:\neg q/p$
$(p \rightarrow r, \{\})$
$(q \rightarrow r, \{\})$

CDL generates two extensions, one containing $(r, \{\neg p, q\})$, the other one $(r, \{\neg q, p\})$, but there is no assertion with supported formula r in all extensions. This is unfortunate since intuitively an assertion (r, X) for an adequate set of formulas X should be contained in the set of skeptical beliefs.

In this paper we further extend the ideas underlying CDL and show how the remaining problems can be solved. Our approach is based on a fundamental shift of the role of defaults within the formalism. In Reiter's logic, defaults are rules used to define extensions of a set of classical facts. The defaults themselves are not objects that can be inferred or reasoned upon. In the approach proposed here defaults become the primary objects of the logical language. We even give up the distinction between facts and defaults. This is possible since facts can be represented as special defaults of the form $true{:}true/p$. Extensions will consist of defaults of the form $true{:}p/q$ which we will call *assumables*. If such an assumable is contained in an extension, i.e., if it is actually assumed, this can intuitively be read as: *q since p is consistent* and, roughly, corresponds to containment of the assertion $(q, \{p\})$ in a CDL extension.

The use of defaults as the elements of the logical language may seem unusual, yet it has several advantages. First of all, we can reason about defaults, not just with defaults. This makes it easy to include different intuitions about the right behaviour of a default logic in an explicit form by specifying adequate inference rules. Secondly, we can show how the mentioned problems of Reiter's logic can be handled adequately. And finally, since dependency information can be conserved in derived defaults this may lead to simplified definitions of extensions, as we will see later in Section 4.

The rest of the paper is organized as follows. In Section 2 we introduce a general framework for defining default logics based on the ideas mentioned above. We show that every default logic within this framework is cumulative. In Section 3 we reconstruct Reiter's logic DL within the framework. More precisely, we define a logic L_{DL} such that the consequents of assumables in L_{DL} extensions correspond exactly to Reiter's extensions. In Section 4 we present a particular instance of the framework, the logic $L1$. We prove various properties of $L1$ and show that the floating conclusions problem is solved and a limited form of reasoning by cases is possible. Section 5 discusses the limitations of reasoning by cases in $L1$. To overcome these limitations we introduce in Section 6 the logic $L2$ in which semi-normal defaults can be used to represent priorities between defaults. We then show how a more general form of reasoning by cases can be performed in $L3$, a further modification of $L2$. Section 7 discusses related work.

2 The Framework

We use Reiter defaults of the form $p{:}\, q_1, \ldots, q_n/r$ as the elements of our language. As in DL, p, q_i and r are classical (closed) first order formulas. We use open variables in defaults to represent all of their ground instances. The set of all defaults, i.e., our language, is denoted DEF. Default theories are sets of defaults. There is no distinct treatment of facts and defaults as in DL: facts can be represented as a special type of defaults of the form $true{:}true/p$. As mentioned earlier defaults of the form $true{:}p/q$ are called assumables. We will often use p as shorthand for the fact $true{:}true/p$ as well as p/q for the assumable $true{:}p/q$.

Extensions will be generated in a stepwise manner. In a first step we apply a set of monotonic inference rules R to the defaults in a default theory T. The closure of T under R, denoted $C_R(T)$, will be called *potential belief set*. In a second step the potential belief set of T is used to define the extensions of T. The extensions are subsets of the closure that have to satisfy certain consistency conditions. This leads to the following definition of a default logic:

Definition 3 *(default logic)*
A default logic is a pair (R, Ext), where

- *R is a set of (monotonic) inference rules on defaults,*
- *$Ext{:}\mathbf{P}(DEF) \mapsto \mathbf{P}(\mathbf{P}(DEF))$ is a function mapping a set of defaults D to a set of subsets of D, the extensions of D.*

We usually are only interested in the value of Ext for sets of defaults which are closed under R. Extensions of the closure of $C_R(T)$ will also simply be called extensions of T. We can introduce a skeptical notion of provability in the usual manner:

Definition 4 *(skeptical provability)*
Let $L = (R, Ext)$ be a default logic, T a default theory. A default $d = p{:}\, q_1, \ldots, q_n/r$ is skeptically derivable from T in L iff

- *$Ext(C_R(T)) \neq \emptyset$ and d is contained in all extensions of T, or*
- *$Ext(C_R(T)) = \emptyset$ and $d \in T$.*

The distinction between the two cases in this definition is necessary to avoid trivial failure of cumulativity in the limiting case where no extension exists. In fact, we can easily prove as a first result that every instance of our framework satisfies cumulativity.

Proposition 1 *(cumulativity)*
Let $L = (R, Ext)$ be a default logic, T a set of defaults, d_1 and d_2 defaults. If d_1 is skeptically derivable from T in L then d_2 is skeptically derivable from $T \cup \{d_1\}$ in L iff d_2 is skeptically derivable from T in L.

Proof: Let d_1 be skeptically derivable from T in L. We have to distinguish two cases:

1) T has no extension

According to definition 5 d_1 must be contained in T, and hence $T = T \cup \{d_1\}$. Proposition 1 therefore is trivially satisfied.

2) T has at least one extension

Since extensions are subsets of $C_R(T)$, d_1 must be contained in $C_R(T)$. Therefore $C_R(T) = C_R(T \cup \{d_1\})$ and T has exactly the same extensions as $T \cup \{d_1\}$. □

3 Reconstruction of DL

To illustrate the expressiveness of this framework we will first show how it can be used to model Reiter's default logic DL. Let us first introduce some terminology:

Definition 5 *Let D be a set of defaults. We define*

$$Just(D) = \{q_i \mid p{:}q_1, \ldots, q_n/r \in D\}$$

$$Cons(D) = \{r \mid p{:}q_1, \ldots, q_n/r \in D\}$$

$$Ass(D) = \{p{:}q_1, \ldots, q_n/r \in D \mid p = true\}.$$

We now define a logic $L_{DL} = (R_{DL}, Ext_{DL})$ in the following way:

Definition 6 *(R_{DL})*
R_{DL} is the following set of rules:

$\mathbf{DL_0}$ $\Rightarrow true{:}true/true$

$\mathbf{DL_1}$ $true{:}q_1, \ldots, q_n/r, r \vdash s \Rightarrow true{:}q_1, \ldots, q_n/s$

$\mathbf{DL_2}$ $true{:}q_1, \ldots, q_n/r, true{:}q_{n+1}, \ldots, q_m/s \Rightarrow true{:}q_1, \ldots, q_m/r \wedge s$

$\mathbf{DL_3}$ $true{:}q_1, \ldots, q_n/r, r{:}q_{n+1}, \ldots, q_m/s \Rightarrow true{:}q_1, \ldots, q_m/s$

DL_0 is necessary to guarantee that tautologies are derivable, for instance if T is empty. DL_1 allows us to generate assumables with weaker consequents. DL_2 generates a form of "conjunction" of two assumables. The chaining rule DL_3 makes it possible to "eliminate" prerequisites if they appear as consequents of assumables. The derived assumable contains the consistency conditions of both defaults used to derive it.

We define the extensions as follows:

Definition 7 *(L_{DL} extensions)*
Let T be a set of defaults, C the closure of T under R_{DL}. E is an L_{DL} extension of T, i.e. $E \in Ext_{DL}(C)$, iff

1. $E \subseteq Ass(C)$

2. for each assumable $d = true{:}q_1, \ldots, q_n/s \in Ass(C)$*:*
$d \in E$ *iff* $\{q_i\} \cup Cons(E)$ *is consistent for all* i $(1 \leq i \leq n)$.

We can prove that the consequents of the assumables in L_{DL} extensions of T coincide with the DL extensions of a corresponding Reiter default theory (D, W).

Proposition 2 *Let* (D, W) *be a Reiter default theory, and*

$$T = D \cup \{true{:}true/p \mid p \in W\}.$$

The function $Cons$ *is a bijective mapping from the set of* L_{DL} *extensions of* T *to the set of* DL *extensions of* (D, W).

Proof: 1) Let E be an L_{DL} extension of T. We first show $Cons(E)$ is a DL extension of (D, W). We introduce some additional notation: Let $ClosL(S)$ denote the closure of a set of defaults S under the inference rules DL_0, DL_1, DL_2. Let $Clos1(S, T)$ denote the set

$$\{true{:}q_1, \ldots, q_m/s \mid true{:}q_1, \ldots, q_n/r \in S, r{:}q_{n+1}, \ldots, q_m/s \in T\}$$

E can be written in the form $\bigcup_{i=0}^{\infty} E_i$ where

$$E_0 = \{true{:}true/p \mid p \in W\}$$
$$E_{i+1} = (ClosL(E_i) \cup Clos1(E_i, T)) \cap E$$

Let $F = Cons(E)$. According to [14, Theorem 2.1] F is an extension of (D, W) iff $F = \bigcup_{i=0}^{\infty} F_i$ where

$$F_0 = W$$
$$F_{i+1} = Th(F_i) \cup DF(F_i, D)$$

where

$$DF(F_i, D) = \{s \mid p{:}q_1, \ldots, q_n/s \in D, p \in F_i, \neg q_i \notin F\}.$$

We show by induction that, for all i, $Cons(E_i) = F_i$, thereby showing that

$$F = Cons(E) = \bigcup_{i=0}^{\infty} Cons(E_i) = \bigcup_{i=0}^{\infty} F_i$$

which proves that F is an extension of (D, W).
Induction base: $Cons(E_0) = F_0$ is obvious.
Induction: Assume $Cons(E_i) = F_i$. we show that
a) $Cons(ClosL(E_i) \cap E) = Th(F_i)$, and
b) $Cons(Clos1(E_i, T) \cap E) = DF(F_i, D)$
a) follows from compactness of first order logic and by the fact that rules DL_0, DL_1

and DL_2 do not introduce defaults with new justifications not already in E_i.
b) Assume $d = true{:}q_1, \ldots, q_m/s$ is in $Cons(Clos1(E_i, T) \cap E)$. There must be $true{:}q_1, \ldots, q_n/r \in E_i$ and $r{:}q_{n+1}, \ldots, q_m/s \in T$. Moreover, since $d \in E$, $Cons(E)$ does not contain $\neg q_i, 1 \leq i \leq n$. By induction hypothesis we have $r \in F_i$. Moreover, $r{:}q_{n+1}, \ldots, q_m/s \in D$, and since $\neg q_i \notin F = Cons(E)$ we have $s \in DF(F_i, D)$.
Vice versa, if $s \in DF(F_i, D)$, then there must be $r \in F_i$ and $d = r{:}q_1, \ldots, q_n/s \in D$, such that $\neg q_i \notin F = Cons(E)$. By induction hypothesis

$$true{:}p_1, \ldots, p_m/r \in E_i.$$

Since $d \in D$ implies $d \in T$ we obtain

$$true{:}p_1, \ldots, p_m, q_1, \ldots, q_n/s \in Clos1(E_i, T).$$

Moreover, since $\neg q_i \notin F = Cons(E)$, this default must be contained in $Clos1(E_i, T) \cap E$ and hence $s \in Cons(Clos1(E_i, T) \cap E)$.

2) Let $F = \bigcup_{i=0}^{\infty} F_i$ be an extension of (D, W). We show that there is an $L1$ extension E of T such that $Cons(E) = F$. We construct E as follows:

$$E_0 = \{true{:}true/p \mid p \in F_0\}$$
$$E_{i+1} = (ClosL(E_i) \cup Clos1(E_i, T)) \setminus F*$$

where $ClosL$ and $Clos1$ are defined as in 1) and $F*$ is the set

$$\{true{:}q_1, \ldots, q_n/r \mid \neg q_i \in F\}.$$

An inductive proof similar to the one in 1) shows that, for all i, $Cons(E_i) = F_i$. Therefore $Cons(E) = F$. It remains to be shown that E is an extension of T.

Again we use C to denote the closure of T under R_{DL}. Obviously $E \subseteq Ass(C)$. Also condition 2) of the definition of Ext_{DL} is satisfied: assumables with justification q_i such that $\neg q_i \in F = Cons(E)$ are excluded from E as a simple induction shows. Moreover, assume there is $d = true{:}q_1, \ldots, q_n/r \in Ass(C)$ such that for all i $(1 \leq i \leq n)$ there is no default with consequent $\neg q_i$ in E. Since then for all i $\neg q_i \notin F$, there must be j such that $d \in E_j$, and hence $d \in E$. Therefore E is an extension of T.

3) To establish bijectivity it remains to be shown that different $L1$ extensions have different images under $Cons$. Let $E1$ and $E2$ be extensions of a set of defaults T. Assume $E1 \neq E2$, but $Cons(E1) = Cons(E2)$. Let $d = true{:}q_1, \ldots, q_n/r$ be a default in $E1 \setminus E2$. Since $d \notin E2$ there must be a q_i such that $\neg q_i \in Cons(E2)$. But since $Cons(E2) = Cons(E1)$ this means that $\{q_i\} \cup Cons(E1)$ is inconsistent and $E1$ is not an extension, contrary to our assumption. □

Although we have an exact correspondence between DL extensions and consequents of L_{DL} extensions the skeptical inference relations of the logics obviously do not coincide in a similar sense. This is partly due to our definition of skeptical derivability, which does not treat non-existence of extensions as inconsistency. It is also due to the floating conclusions problem. A formula p which is contained in all DL extensions may have different justifications in different L_{DL} extensions. Thus there may be no q such that $true{:}q/p$ is in all L_{DL} extensions.

Our result also nicely shows the implicit assumptions underlying DL that lead to failure of cumulativity. DL cannot distinguish between situations where p is believed independently, and situations where p is a default conclusion depending on the consistency of certain other formulas. L_{DL}, by keeping track of consistency assumptions in the justifications of derived defaults, marks this distinction and thus can satisfy cumulativity. Consider the reformulation of Makinson's original example showing the failure of cumulativity for DL [9]:

$$\{true{:}p/p, p \vee q{:}\neg p/\neg p\}$$

The closure contains

$$true{:}p/p \vee q$$

(application of DL_1 to the first default). This default is also contained in the single L_{DL} extension, i.e. we derive: $p \vee q$ since p is consistent. Adding this default as a premise does no harm. DL, on the other hand, can only derive $p \vee q$ in this example, that is a formula that corresponds to the default

$$true{:}true/p \vee q$$

The addition of this default clearly changes the generated extensions and thus the skeptical inference relation.

Of course, remodelling Reiter's logic is not the main purpose of our new framework. In the next section we will define a new and somewhat more interesting default logic.

4 The Default Logic $L1$

We now define the default logic $L1$. To do this we have again to define a set of rules R_{L1} and the mapping Ext_{L1}. For the rest of the paper we will restrict ourselves to defaults with single justifications.

Definition 8 *(R_{L1})*
R_{L1} consists of the following inference rules:

R0 *tautologies:*
$\Rightarrow true{:}true/true$

R1 *weakening:*
$true{:}p/q, q \vdash r \Rightarrow true{:}p/r$

R2 *combination:*
$true{:}p_1/q_1, true{:}p_2/q_2 \Rightarrow true{:}p_1 \wedge p_2/q_1 \wedge q_2$

R3 *chaining:*
$true{:}p/q, q{:}r/s \Rightarrow true{:}p \wedge r/s$

R4 *equivalence:*
$p{:}q/r, s \equiv p, m \equiv q \Rightarrow s{:}m/r$

R5 *disjoining:*
$true{:}p/q, true{:}r/q \Rightarrow true{:}p \vee r/q$

R6 *cases:*
$p{:}q/r, s{:}m/n \Rightarrow p \vee s{:}q \wedge m/r \vee n$

$R0$ to $R3$ are basically single justification versions of the rules DL_0 to DL_3 we used in R_{DL}. The important difference between L_{DL} and $L1$ will be that in the new logic we enforce joint consistency of all justifications. It is therefore possible to replace multiple justifications by their conjunction.

$R4$ makes it possible to replace prerequisites and justifications by equivalent formulas.

$R5$ allows us to disjoin the justifications of two assumables with the same consequent. We will see later that this rule solves the floating conclusions problem in our approach.

$R6$ makes a (limited) form of reasoning by cases possible. Consider one of the standard examples: from $emu{:}runs/runs$ and $ostrich{:}runs/runs$ we infer using $R6$ and simplifying the new justification using $R4$

$$emu \vee ostrich{:}runs/runs$$

which in turn yields $true{:}runs/runs$ if $true{:}true/emu \vee ostrich$ is given.

We now define the second part of $L1$, Ext_{L1}. We first need another auxiliary definition.

Definition 9 *(kernel)*
Let S be a set of defaults. The kernel of S is the set

$$Kern(S) = \{d \in S \mid d = true{:}true/p\}.$$

Definition 10 *($L1$ extension)*
Let T be a default theory, C its closure under R_{L1}. E is an extension of T, i.e. $E \in Ext_{L1}(C)$, iff E is a maximal subset of C such that

1. *$E \subseteq Ass(C)$, that is E consists of assumables in C,*

2. *$Kern(C) \subseteq E$,*

3. *$Just(E) \cup Cons(E)$ is consistent.*

This definition is, contrary to Reiter's original one and the one used for CDL in [2], not based on a fixed point construction. This is possible since the derived defaults carry sufficient dependency information with them. The reader should also note that an assumable d of the form $true{:}p/q$ gets a stronger meaning when it is contained in an extension, i.e. when it actually is assumed, than when it merely is a member of the default theory T. In T its intuitive reading is: *q if p is consistent* (with what is believed and the justifications of other believed formulas). If d is contained in an extension, then we know that p must be consistent and hence can strengthen the intuitive reading to: *q since p is consistent.*

Let us consider some examples. Remember that we use p/q as an abbreviation for $true{:}p/q$ and q as an abbreviation for $true{:}true/q$. As usual, we first have to show whether Tweety flies:

Example 1: Tweety

1) $Peng(Tw)$
2) $Bird(Tw)$
3) $Bird(Tw){:}Flies(Tw)/Flies(Tw)$

We apply $R3$ to 2) and 3) and obtain, after simplification of the justification using $R4$, the assumable $Flies(Tw)/Flies(Tw)$. This assumable is contained in the single extension, that is we derive: Tweety flies, since it is consistent to assume that Tweety flies.

Adding

4a) $Peng(Tw) \rightarrow \neg Flies(Tw)$

eliminates $Flies(Tw)/Flies(Tw)$, which is still contained in the closure, from the single extension. The reason is that by applying the combination rule $R2$ to 1) and 4a) and by applying the weakening rule $R1$ to the result we obtain $true{:}true/\neg Flies(Tw)$. This assumable belongs to the kernel, must therefore be contained in the extension, and thus excludes $Flies(Tw)/Flies(Tw)$.

If instead of 4a we add

4b) $Peng(Tw){:}\neg Flies(Tw)/\neg Flies(Tw)$

we get two extensions, one containing

$$Flies(Tw)/Flies(Tw),$$

the other one

$$\neg Flies(Tw)/\neg Flies(Tw).$$

The next example has been used in [13] to illustrate DL's problems with mutually inconsistent justifications of defaults.

Example 2: Broken arms

1) $usable(x) \wedge \neg broken(x)/usable(x)$
2) $broken(left) \vee broken(right)$

We obtain two extensions. The first extension contains

$$usable(left) \wedge \neg broken(left)/usable(left)$$

Read: the left arm is usable since it is consistent that it is usable and not broken. The second extension contains

$$usable(right) \wedge \neg broken(right)/usable(right)$$

Contrary to DL, $L1$ produces no extension in which both arms are usable.

The next example is also taken from [13]. It shows that a limited form of reasoning by cases is possible in $L1$:

Example 3: Reasoning by cases

1) $emu{:}runs/runs$
2) $ostrich{:}runs/runs$
3) $emu \vee ostrich$

We apply $R6$ to 1) and 2), and $R3$ to the result and 3). Application of $R4$ leads to the derivation of $runs/runs$ in the closure. This default is also contained in the single extension, that is $runs$ is derivable since it is consistent with what is derivable.

Example 4: Floating conclusions [4]

1) $\neg a/b$
2) $\neg b/a$
3) $a \rightarrow c$
4) $b \rightarrow c$

We can apply $R3$ to 1) and 4) to obtain (after equivalence transformations using $R4$)

5) $\neg a/c$

Similarly, we apply $R3$ to 2) and 3) to obtain

6) $\neg b/c$

From 5) and 6) we derive using $R5$

7) $(\neg a \vee \neg b)/c$

We obtain two extensions, the first one contains 1), 3), 4), 5) and 7), the second one 2), 3), 4), 6) and 7). Since 7) is contained in both extensions c with a proper justification is derivable, and the floating conclusions problem does not appear.

The reader may have noticed that the defaults in all but the last of the examples used above either are "facts" of the form $true{:}true/p$, or semi-normal defaults of the form $p{:}q/r$, where $q \vdash r$. This is not incidental. We will later show that we can prove some nice properties of $L1$ if T consists of defaults of these two types. On the other hand, if we admit arbitrary defaults then the rules of $L1$ may lead to unwanted effects. In particular, it may be the case that an assumable a_1 depending on assumable a_2 may be contained in an extension although a_2 is not. Consider the default theory

1) $true{:}p/\neg p$
2) $\neg p{:}q/q$

Applying $R3$ we derive

3) $true{:}p \wedge q/q$

3) is contained in the single extension. It is, however, questionable whether 3) should actually be considered derivable. There seems to be something wrong with 1), a default which was needed to derive 3), and 1) is not contained in the extension.

To avoid such effects we might come up with a more complicated set of inference rules that take the peculiarities of non-semi-normal defaults into account. Since there seems to be no good reason for using such defaults in the first place, however, we will stick to our simpler set of rules and assume that all defaults in T are of a reasonable form. We therefore define what we mean by semi-normal default theories:

Definition 11 *(T-semi-normal default)*
Let T be a set of defaults. A default $d = p{:}q/r$ is called T-semi-normal iff there is a set of defaults $true{:}true/s_i$ $(1 \leq i \leq n)$ in T such that

$$\{q, s_1, \ldots, s_n\} \vdash r$$

Note that all "facts" $true{:}true/p \in T$ are T-semi-normal.

Definition 12 *(semi-normal default theories)*
A set of defaults T is a semi-normal default theory iff every default in T is T-semi-normal.

We will next prove some results which show that $L1$ is quite well-behaved for semi-normal default theories. We first prove the following useful lemma:

Lemma 1 *Let T be a semi-normal default theory, C the closure of T under the rules R_{L1}. A default d is contained in C only if d is T-semi-normal.*

Proof: We prove the lemma by induction on the length l of the shortest proof of $p{:}q/r$ from T. For $l = 0$ the lemma is trivially satisfied. Assume the lemma holds for defaults whose shortest proof is of length $n-1$. Inspection of the rules in R_{L1} shows that in each case the lemma is satisfied for the derived default. As an example consider $R5$:

$$true{:}p/q, true{:}s/q => true{:}p \vee s/q$$

By induction hypothesis there is a set of defaults

$$true{:}true/p_j$$

in T such that $\{p, p_1, \ldots, p_n\} \vdash q$. Similarly, there is a set of defaults $true{:}true/s_k$ in T such that $\{s, s_1, \ldots, s_m\} \vdash q$. It follows from the rules of propositional logic that

$$\{p \vee s, p_1, \ldots, p_n, s_1, \ldots, s_m\} \vdash q$$

Hence the derived default satisfies the lemma.
The proof proceeds in a similar way for all other rules. □

The lemma implies that adding a derivable default d to a semi-normal theory T preserves semi-normality. The following is an immediate corollary of the lemma

Corollary 1 *Let T be a semi-normal default theory, C the closure of T under the rules R_{L1}. A default true:true/r is contained in C only if there is a set of defaults true:true/s_i $(1 \leq i \leq n)$ in T such that*

$$\{s_1, \ldots, s_n\} \vdash r$$

Proposition 3 *(existence of extensions)*
Let T be a default theory, C its closure under R_{L1}. T has at least one extension iff $Cons(Kern(C))$ is consistent.

Proof: "$\Leftarrow$"
Let K be the kernel of C. Let

$$(true{:}j_1/c_1, true{:}j_2/c_2, \ldots)$$

be an arbitrary enumeration of the assumables in C. We construct an extension E as follows: $E = \bigcup_{i=0}^{\infty} E_i$, where

$E_0 = K$, and
$E_{i+1} =$ if $E_i \cup \{j_{i+1}, c_{i+1}\}$ consistent then $E_i \cup \{true{:}j_{i+1}/c_{i+1}\}$ else E_i.

E consists of assumables in C and contains $Kern(C)$. Furthermore, since $Cons(K)$ is consistent, by construction $Just(E) \cup Cons(E)$ is consistent and E is a maximal

set satisfying this condition. E therefore is an extension of T.
"$\Rightarrow$" obvious $\Box$

As an immediate consequence of Proposition 3 and Corollary 1 we can characterize the semi-normal default theories possessing extensions without having to determine the closure first:

Corollary 2 *(existence of extensions for semi-normal theories)*
Let T be a semi-normal default theory. T has at least one extension iff $Cons(Kern(T))$ is consistent.

The following proposition is of interest from the computational point of view:

Proposition 4 *(semi-monotony)*
Let T be a semi-normal default theory, $d = p{:}q/r$ a default such that $q \vdash r$ and q is not equivalent to true. For every extension E of T there is an extension E' of $T' = T \cup \{d\}$ such that $E \subseteq E'$.

Proof: Follows from the fact that the addition of d lets the closure under R_{L1} grow monotonically, i.e., $C_{R_{L1}}(T) \subseteq C_{R_{L1}}(T')$, and the fact that according to Corollary 1 the kernel remains unchanged, i.e., $Kern(C_{R_{L1}}(T)) = Kern(C_{R_{L1}}(T'))$ $\Box$

The examples and results show that $L1$ is an interesting nonmonotonic logic in which some of the problems of DL mentioned in Section 1 are avoided. Nevertheless one can - and in fact we will - consider alternative choices for rules and even for the definition of extensions. As mentioned earlier, we do not believe that there is *the* single perfect nonmonotonic logic. As in modal logic where different intuitions about the correct meaning of the modalities lead to different modal systems there are different interesting nonmonotonic logics with different properties and different applications. This is why we have in this paper focussed on a general framework that makes it easy to specify different intuitions about nonmonotonic reasoning.

5 Limitations of reasoning by cases in $L1$

Let us now try to motivate why one might consider alternative definitions of a non-monotonic logic as well. We have shown in the last section that a limited form of reasoning by cases is possible in $L1$. Unfortunately, some more general cases cannot be handled adequately in $L1$. Consider the following example:

1) $Italian{:}Black/Black$
2) $German{:}Blond/Blond$
3) $\neg Blond \vee \neg Black$
4) $German \vee Italian$

Applying $R6$ to 1) and 2) does not help us to obtain a default with consequent $Blond \vee Black$ in any extension, since the justification of the derived rule, $Black \wedge Blond$, is inconsistent with 4). The justification of the derived default simply is too strong in this case. Can we come up with a weaker justification for the derived default that leads to the desired results? What if we change $R6$ to $R6'$:

$$p{:}\,q/r, s{:}\,m/n \Rightarrow p \vee s{:}\,q \vee m/r \vee n$$

Unfortunately, now the justification $q \vee m$ in the derived default is too weak. In our hair-color example, for instance, $R6'$ would lead to the unwanted derivation of $Black \vee Blond/Black$ if additionally

5) $German \wedge \neg Blond$

is given.

What is needed in this example to get the right answers are two separate consistency checks (for $Black$ and for $Blond$) instead of a single check (for $Black \wedge Blond$). Assume $Black$ and $Blond$ are consistent. We then know that if the person is Italian we can derive that her hair is black, and if she is German we can derive it is blond. In both cases we derive that it is black or blond.

$L1$ does not admit multiple consistency checks. This problem can be solved by reintroducing defaults with multiple justifications. However, this is not a straightforward extension since we have to be careful not to run into problems of joint consistency of justifications again (broken arms example). We also want to avoid an explosion of the number of necessary justifications.

We will therefore propose a solution which does not involve multiple justifications in the next section. This solution only works after some modifications of the logic, however. The modification will, among other things, concern the treatment of implicit priorities in defaults. [2]

[2] In a recent paper, Delgrande and Jackson address the problem of reasoning by cases in DL and claim to have solved it in a modified version of DL called PJ-logic [5]. Their solution is unsatisfactory for the following reasons: they use prerequisite-free defaults and represent a default "A's are typically B" for which contraposition is unwanted as $true{:}\,B(x)/A(x) \supset B(x)$. First of all, such a default is not equivalent to $A(x){:}\,B(x)/B(x)$ and the restriction to prerequisite-free defaults leads to a loss of expressiveness. More importantly, our hair color example is not handled correctly. The defaults become $true{:}\,Black/Italian \supset Black$ and $true{:}\,Blond/German \supset Blond$. Delgrande and Jackson require joint consistency of justifications of applied defaults. Since $Black \wedge Blond$ is inconsistent they never can apply both defaults in the same extension and thus are unable to derive $Black \vee Blond$ from $Italian \vee German$.

6 Implicit Priorities: The Logic $L2$

Consider the two defaults

1) $true{:}p/p$
2) $true{:}\neg p \wedge q/q$

In Reiter's logic only one extension is generated, namely $Th(\{p\})$. Default 2) is not applied since the consequent of 1) blocks 2). On the other hand, the consequent of 2) cannot block 1). Thus 1) implicitly gets priority over 2).

In $L1$ the ability to use semi-normal defaults for encoding priorities is lost. In our example we get two extensions, that is 2) may be preferred to 1). The reason is that in $L1$ not only consequents, but also justifications have the power to block conflicting defaults.

As shown by Reiter and Criscuolo [15] it is sometimes useful to keep the ability of expressing default priorities via semi-normal defaults. We will therefore define a logic $L2$ in which such priorities are preserved. $L2$ will be based on a new definition of extensions. The intuition we want to capture is that in the example we just discussed only one extension should be generated, i.e. 1) should be able to eliminate 2), but not vice versa. On the other hand we still want to get two extensions in examples like the following:

1) $true{:}r \wedge p/p$
2) $true{:}\neg r \wedge q/q$

We therefore have to treat justifications and consequents differently in the definition of extensions. The following definition captures our intuitions:

Definition 13 *($L2$ extensions)*
Let T be a default theory, C its closure under R_{L1}. E is an $L2$ extension of T iff

1. *E is an $L1$ extension, and*
2. *for each assumable $true{:}p/q \in C$:*
   $true{:}p/q \notin E$ only if
   - *$\{p\} \cup Cons(E)$ inconsistent, or*
   - *$\{q\} \cup Just(E) \cup Cons(E)$ consistent.*

The idea underlying this definition is the following: an $L1$ extension E is only accepted as an $L2$ extension if there is no assumable $d = true{:}p/q$ that has been excluded from E just because its consequent is inconsistent with the justifications of included assumables. We only accept two reasons for excluding d from an $L2$ extension. One reason is that p is inconsistent with the consequents in E. If p is not inconsistent with the consequents yet d is not in E, then since E is an $L1$ extension p must be inconsistent with $Just(E) \cup Cons(E)$ given T is semi-normal. [3] In this case we only accept the exclusion of d from E if q is not itself inconsistent with the consequents and justifications, since otherwise we expect the consequent to override the conflicting justifications.

We will use our earlier examples to illustrate that this definition handles the implicit priorities correctly. Consider first the example

1) $true{:}p/p$
2) $true{:}\neg p \wedge q/q$

It is not difficult to see that the unwanted extension E_1 containing 2) disappears since, in terms of our informal discussion of Definition 13, 1) is excluded without good reason: the justification of 1) is not inconsistent with $Cons(E_1)$, and the consequent of 1) is not consistent with $Just(E_1) \cup Cons(E_1)$. The exclusion of 1) therefore is unwarranted. On the other hand, the second $L1$ extension, E_2, contains 1) and is an $L2$ extension as this time the exclusion of 2) is sanctioned by Definition 13.

We also obtain the desired results in our second example:

1) $true{:}r \wedge p/p$
2) $true{:}\neg r \wedge q/q$

Again we get two $L1$ extension. Both are $L2$ extensions. Consider E_1, the extension containing 1), but not 2). The exclusion of 2) is sanctioned since the second alternative in Definition 13 applies. The second $L1$ extension is an $L2$ extension for symmetrical reasons.

Unfortunately, the (re)introduction of implicit priorities destroys, like in DL, the existence of extensions for semi-normal theories. The following example was originally used in [15] to show non-existence of extensions for semi-normal DL theories and can be used to illustrate this:

1) $true{:}\neg a \wedge b/b$
2) $true{:}\neg b \wedge c/c$
3) $true{:}\neg c \wedge a/a$

In $L1$ three extensions are obtained, each of which is generated by one of the premises. However, none of these is an $L2$ extension. Consider as an example the $L1$ extension E_1 containing 1). It is not difficult to see that this extension is not an $L2$ extension since 3) is excluded for no acceptable reason according to Definition 13. The two other $L1$ extensions are rejected for similar reasons.

The non-existence of $L2$ extensions for semi-normal default theories is not too surprising. $L2$ is expressive enough for representing priorities between defaults, and

[3] Again we do not explicitly forbid using default theories which are not semi-normal. We are, however, mainly interested in the behaviour of our definitions for semi-normal theories.

nothing prevents us from specifying cyclic priorities, as in our example, which lead to situations where no $L1$ extension is an $L2$ extension. [4]

In the rest of this section we show how $L2$, more precisely a further modification of $L2$, can be used for reasoning by cases. We have seen in the last section that, except in the most simple cases, reasoning by cases seems to require multiple consistency checks. As argued earlier we want to avoid the introduction of multiple justifications as this might interfere with our joint consistency requirement, and it might lead to an explosion of the number of justifications needed. Instead we use a trick to avoid multiple justifications (a similar idea has recently been proposed by U. Junker, personal communication).

We use new propositional constants C_p to express that p is consistent. To keep things simple we will not allow nesting of C, that is p itself has to be one of the original formulas not containing C. We will limit the use of these constants so as to guarantee their intended meaning: the only positive appearances of C_p will be in justifications of defaults.

Given these additional constants we replace $R6$ by a rule schema that allows us to derive:

$$p_1 \vee \ldots \vee p_n : C_{q_1} \wedge \ldots \wedge C_{q_n} \wedge (r_1 \vee \ldots \vee r_n) / r_1 \vee \ldots \vee r_n$$

from the defaults

$$p_1 : q_1 / r_1, \ldots, p_n : q_n / r_n$$

whenever all q_i do not contain C.

We further have to make provisions that an assumable with C_p occurring positively in its justification is not contained in an extension E whenever $Just(E) \cup Cons(E) \vdash \neg p$. Note that including the default

$$\neg p : \neg C_p / \neg C_p$$

is not sufficient, since this only takes the consequents in E into account.

The easiest way to achieve the desired effect is to change the monotonic logic underlying the definition of $L2$ extensions. Let us define $L3$ extensions in the same way as $L2$ extensions, but with a stronger notion of consistency. Instead of classical consistency we use consistency in a stronger logic, the logic obtained by adding to classical logic the inference rule

$$\neg p \Rightarrow \neg C_p$$

Note that this technique only works as intended if we base our definition of $L3$ extensions on the priority preserving definition of $L2$ extensions, not on $L1$ extensions.

[4]Probably the easiest way to guarantee existence of extensions would be the following: define $L2'$ extensions of T as $L2$ extensions of T', where T' is a maximal subset of T for which an $L2$ extension exists. The existence of $L2'$ extensions is, obviously, guaranteed.

It is essential that positive occurrences of C_p in a justification are overridden if $\neg p$ is derivable and cannot be used to block the derivation of $\neg p$.

Reconsider our hair-color example. From:

1) $Italian{:}Black/Black$
2) $German{:}Blond/Blond$

we derive

$$Italian \vee German{:}C_{Black} \wedge C_{Blond} \wedge (Black \vee Blond)/Black \vee Blond$$

If we have the additional premise

3) $Italian \vee German$

we get, as intended, one extension containing the assumable

$$C_{Black} \wedge C_{Blond} \wedge (Black \vee Blond)/Black \vee Blond$$

If we add

4) $\neg Blond$

then the assumable with consequent $Black \vee Blond$ will not be contained in the single $L2$ extension since its justification containing C_{Blond} is overridden by $\neg Blond$. This is exactly what we expect from reasoning by cases since the person at hand may be a German and there is no reason to conclude that her hair is black.

Obviously, there are many further alternative ways to define a default logic. For instance, we might redefine the notion of kernel and require that some assumables in a default theory T whose justifications are not equivalent to $true$ be included in the extensions as well, i.e.,

$$Kern(T) = S$$

where S is a distinguished subset of defaults $true{:}p/q$ of T. Every extension then has to be consistent with $Just(S) \cup Cons(S)$. This can, for instance, be used to model Poole-systems with constraints [12] or CDL [2] in our framework. Another idea would be to use explicit priority orderings of defaults instead of the implicit orderings induced by semi-normal defaults. The reader should be convinced by now that our framework makes it quite easy to design logics which possess these or other properties which may be desired for some particular application.

7 Related Work

The research reported in this paper was originally motivated by some defects in Reiter's DL. A first solution, CDL, was very close to DL but used assertions as elements of the language (which is already close to using defaults). CDL solved some of the problems but introduced a new one, the floating conclusions problem, and left some others unsolved. In this paper we presented a framework for defining cumulative logics where

- defaults are the elements of the logical language,
- inference rules on defaults generate a potential belief set,
- extensions are defined as subsets of the potential belief set satisfying certain maximality and consistency conditions.

We proposed several logics fitting into this framework. The logic $L1$ solved the floating conclusions problem and made it possible to perform reasoning by cases in simple examples. No fixed point construction was needed to define the extensions. We proved that $L1$ is semi-monotonic for semi-normal default theories and that extensions exist whenever the "hard" facts are consistent in a natural sense. In the logic $L2$ semi-normal defaults can be used to express implicit priorities between defaults. $L3$ was obtained from $L2$ by strengthening the underlying notion of consistency and adding a further rule schema. In $L3$ reasoning by cases in an unrestricted form is possible.

Many ideas underlying this approach are not entirely new, and the presented work has benefitted from various sources. Gelfond and colleagues [6], for instance, have considered facts as special kinds of defaults in their variant of DL. They have not considered using inference rules on defaults, however. There is also a close relationship between our framework and earlier work in the area of belief revision, in particular [10]. These authors, however, focus on monotonic dependencies, whereas we take nonmonotonic dependencies into account.

Another related area is the work on conditional approaches to nonmonotonic reasoning, in particular the work of Kraus, Lehmann and Magidor [7] [8]. Here inference rules on conditionals are used to formalize cumulative inference relations. Using defaults instead of conditionals gives us an additional "degree of freedom" in the definition of default logics. This makes it, for instance, possible to model Reiter-like default logics without giving up cumulativity.

The presented work is probably closest in some of its basic intuitions to Nait Abdallah's ionic logic [11]. This logic is based on so-called ions, that is structures of the form $(X, p)_*$ which directly correspond to our assumables. Ionic logic allows for arbitrary nesting of ions and is more general than our approach in this respect. The rules presented in [11] have some similarity to our $L1$ rules, and the theorems of ionic logic roughly correspond to our potential belief set. However, Nait Abdallah abandons nonmonotonicity and thus seems to throw out the baby with the bath water. Ionic logic is monotonic and yields hypothetical conclusions of the form "p if q is consistent" only. The burden of checking whether q actually is consistent (in whatever sense) is left to the user of the logic.

One aspect that distinguishes our approach from the related work discussed above is our emphasis on a *framework*. As mentioned several times in this paper we strongly believe that there is not *the* single best nonmonotonic logic, as there is not *the* single best modal logic. There are different intuitions about how a nonmonotonic logic should behave, and there are applications that may require different properties of a logic. What seems to be important, then, is to have a framework that supports the design of a particular logic. We hope to have demonstrated that our framework is useful for that purpose.

References

[1] Bidoit, Nicole, Froidevaux, Christine, Negation by Default and Non-Stratifiable Logic Programs, Research Report 437, Universite Paris Sud, Centre d'Orsay, LRI, 1988

[2] Brewka, Gerhard: Cumulative Default Logic - In Defense of Nonmonotonic Inference Rules, Artificial Intelligence 50, 1991

[3] Brewka, Gerhard, Assertional Default Theories, Proc. ECSQAU, Marseille 91, Springer Verlag, 1991

[4] Brewka, Gerhard; Makinson, David; Schlechta, Karl, Cumulative Inference Relations for JTMS and Logic Programming, Proc. First International Workshop on Logic Programming and Non-Monotonic Reasoning, Washington, 1991, full version to appear in Proc. NIL-90, Workshop on Nonmonotonic and Inductive Logics, Karlsruhe 1990, 1991

[5] Delgrande, J.P., Jackson, W.K., Default Logic Revisited, Proc. Second International Conference on Principles of Knowledge Representation and Reasoning, 1991

[6] Gelfond, M., Lifschitz, V., Przymusinska, H.,Truszczynski, M., Disjunctive Defaults, Proc. Second International Conference on Principles of Knowledge Representation and Reasoning, 1991

[7] Kraus, S., Lehmann, D., Magidor, M., Nonmonotonic Reasoning, Preferential Models and Cumulative Logics, Artificial Intelligence, 1990

[8] Lehmann, D., What Does a Conditional Knowledge Base Entail?, Proc. First International Conference on Principles of Knowledge Representation and Reasoning, 1990

[9] Makinson, David, General Theory of Cumulative Inference, Proceedings 2nd International Workshop on Non-Monotonic Reasoning, Grassau, Germany, Springer, LNAI 346 (1989) 1-18.

[10] Martins, Joao; Shapiro, Stuart C., A Model for Belief Revision, Artificial Intelligence 35, 1988

[11] Nait Abdallah, Areski, Syntax and Semantics of a Monotonic Framework for Non-monotonic Reasoning, Univ. of Western Ontario, London, Ontario, Dep. of Comp. Science, Report Nr. 293, 1991

[12] Poole, David, A Logical Framework for Default Reasoning, Artificial Intelligence 36 (1988) 27-47.

[13] Poole, David, What the Lottery Paradox Tells Us About Default Reasoning, Proceedings First International Conference on Principles of Knowledge Representation and Reasoning, Toronto (1989) 333-340.

[14] Reiter, Raymond, A Logic for Default Reasoning, Artificial Intelligence 13 (1980) 81-132.

[15] Reiter, Raymond, Criscuolo, G., On Interacting Defaults, Proc. IJCAI 81, 1981

[16] Schaub, Torsten, Assertional Default Theories: A Semantical View, Proc. Second International Conference on Principles of Knowledge Representation and Reasoning (KR-2), 1991