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be deduced from Arrow's paradox. Indeed, in such a crisp context we can simply conclude that only with two alternatives we can find (absolute) rationality. In our model, we have seen how the number of possible rules decreases with the number of alternatives from the case in which there are only two alternatives where no rule leads to absolute irrationality, to the general case with an unknown number of alternatives, where we have proven the existence of rules that never lead to irrationality.

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For any rational cycle

$$C \equiv x_1 P_1 x_2 P_2 \cdots x_k P_k x_{k+1} P_{k+1} x_1$$

two possibilities arise:

1. $P_{k+1} \in \{W, B\}$ then since $\mu_I(x_{k+1}, x_1) = 1$ it is clear that $\mu_B(x_{k+1}, x_1) = \mu_W(x_{k+1}, x_1) = 0$. Therefore, the weight associated to C is zero.

2. $P_{k+1} = I$ then

$$C' \equiv x_1 P_1 x_2 P_2 \cdots x_k P_k x_1$$

is a rational cycle. Moreover, since $\mu_I^i(x_{k+1}, x_1) = 1$ for all $i = 1, 2, \ldots, n$, $\mu_I(x_{k+1}, x_1) = 1$. Therefore, the weight of C is equal to the weight of C' which is zero since $A_\mu(G) = 0$.

Summing up, $A_\mu(G') = 0$ and the theorem is proven. ■

6 Final Comments

It must be pointed out once again that the aggregation model here developed is restricted to aggregation rules that can be represented by a unique intensity aggregation rule, once conditions IIA, UD and N have been assumed. As a direct consequence we have that these aggregation rules are defined without taking into account the number of alternatives. Therefore, Non Absolutely Irrational (NAI) rules are those rules associated to intensity aggregation rules that will never lead to absolute irrationality. A characterization of these NAI rules has been given. By means of this characterization, the non irrationality of many rules can be checked quite easily. Notice in particular the importance of Corollary 4.2. Any complete rule satisfying positive responsiveness, that is any complete rule which is sensitive to individual changes, is automatically a non absolutely irrational intensity aggregation rule.

Moreover, it is obvious that if a given rule leads to k-element chains with zero degree of rationality, such a rule can never be considered in aggregation problems with more than k alternatives. In addition, notice that from example 5.1 and theorem 5.1 we can infer that ethical rules leading to k-element chains with zero degree of rationality will also lead to longer chains with zero degree of rationality. This shows that the number of alternatives makes in fact more complex (though still solvable) the aggregation problem. This fact is very well known from practice and it cannot

2. $|B_{(x,y)}| \leq 1$ and $|W_{(x,y)}| \geq 4$. This case is analogous to the previous one.

3. $|B_{(x,y)}| \geq 2$ and $|W_{(x,y)}| \geq 2$.

   Once again let us consider the cycles either of type $xByWzPx$ or of type $xByPzWx$ and suppose that $\mu_W(y,z) = \mu_W(z,x) = 0$.

   We then have,

$$|B_{(x,y)}| \geq 2 \ , \ |W_{(x,y)}| \geq 2 \ , \ |W_{(y,z)}| \leq 1 \ , \ |W_{(z,x)}| \leq 1 \ .$$

   Since we have to put five elements in the W-sets it must be $|W_{(x,y)}| \geq 3$. Three of the elements in $W_{(x,y)}$ must also belong either to $B_{(y,z)}$ or to $B_{(z,x)}$ for otherwise they would not be rational. Therefore, either $B_{(y,z)}$ or $B_{(z,x)}$ has two elements, which lets us conclude that μ is non irrational.

□

Generalizing example 5.1, it can be analogously proven that k-element chains do not lead to irrationality of the rule ψ when the number of individuals is $n \geq 2k-1$.

The example above suggests that, in general, aggregation rules for which there are k-element chains with zero degree of rationality, will lead to longer chains with zero degree of rationality. This is consistent with our intuition that decision problems become more complex as the number of alternatives under consideration gets larger.

THEOREM 5.1 *Given $k \geq 3$, if an intensity aggregation rule $\phi : [0,1]^n \to [0,1]$ satisfying $\phi(1,\ldots,1) = 1$ leads to zero degree of rationality for k-element chains, it does so for $(k+1)$-element chains.*

Proof. Denote as usual with μ the aggregation rule associated to ϕ. Let us consider a chain with k alternatives $G = (x_1 - x_2 - \cdots - x_k - x_1)$, such that $A_\mu(G) = 0$ for some $\mu^1, \ldots, \mu^n$ non absolutely irrational preference relations. Consider now the chain

$$G' = (x_1 - x_2 - \cdots - x_k - x_{k+1} - x_1)$$

and put for all $i = 1, \ldots, n$

$$\mu^i(x_k, x_{k+1}) = \mu^i(x_k, x_1) \ , \ \mu^i(x_{k+1}, x_k) = \mu^i(x_1, x_k) \ , \ \mu^i(x_1, x_{k+1}) = \mu^i(x_{k+1}, x_1) = 1.$$

(b) there exist a pair in the chain (without loss of generality we can suppose that such a pair is (x, y)) such that for all $i = 1, 2, \ldots, n$

$$\mu^i(x, y) + \mu^i(y, x) = 1$$

Therefore, for all $i = 1, 2, \ldots, n$, $\mu^i_I(x, y) = 0$ and $\mu^i_B(x, y) + \mu^i_W(x, y) = 1$.

To the pair (x, y) we associate two sets $W_{(x,y)}$ and $B_{(x,y)}$ which represent respectively the set of individuals j such that $\mu^j_W(x, y) > 0$ and the set of individuals i such that $\mu^i_B(x, y) > 0$. Analogously for the other pairs (y, z) and (z, x).

Notice that if any of the W [resp. B] sets has cardinality at least two then the aggregated opinion μ_W [resp. μ_B] computed on the corresponding pair, will have positive weight.

Three cases are possible:

1. $|B_{(x,y)}| \geq 4$ and $|W_{(x,y)}| \leq 1$.

   The above hypothesys implies that for at least four inviduals i, $\mu^i_B(x, y) = 1$ and $\mu^i_W(x, y) = 0$.

   Let us prove that either a cycle of type $xByWzPx$ or of type $xByPzWx$ must have positive weight, where $P \in \{I, W, B\}$. Notice that if either $\mu_W(y, z)$ or $\mu_W(z, x)$ have positive weight we are done.

   Suppose by contradiction that $\mu_W(y, z) = \mu_W(z, x) = 0$. Therefore, for at least four individuals j, it must be $\mu^j(y, z) = 1$ and $\mu^j_W(y, z) = 0$; and for at least four individuals l it must be $\mu^l(z, x) = 1$ and $\mu^l_W(z, x) = 0$.

   From the hypotheses we have

   $$|B_{(x,y)}| \geq 4 \ , \ |W_{(x,y)}| \leq 1 \ , \ |W_{(y,z)}| \leq 1 \ , \ |W_{(z,x)}| \leq 1 \ .$$

   Since all individuals are not irrational and for each individual the indifference cycle has weight zero, there must be at least 5 cycles with strict preferences whose weight is positive. Each of these five cycles must contain both a W and a B in order to be rational. So, for each individual we must put at least one element in a B-set and at least one element in a W-set and this two sets must correspond to different pairs. On the other hand, the total number of elements that we can put in W-sets is at most 3. We then reach a contradiction.

Finally, it must be noticed that in proving the necessity part of theorem 4.2 we needed long chains. Indeed, when condition (ii) does not hold, irrationality is proven for long chains with more alternatives than individuals. Such a characterization does not hold if the number of alternatives has been a priori fixed and it is smaller than n, as it will be shown in the following example. This fact may suggest a search for additional assumptions leading to rules where short chains do not produce absolute irrationality for the aggregated values.

However, in this paper we did not want to impose any kind of restriction on the number of alternatives, so that an aggregation rule is considered to be non irrational if and only if no chain of any length can produce irrationality.

Example 5.1 Let us consider a group with $n \geq 5$ people and the following intensity aggregation rule:

$$\psi(a^1, \ldots, a^n) = \begin{cases} \frac{\sum_{i=1}^n a^i}{n} & \text{if } |\{i : a^i = 1\}| < n-1 \\ 1 & \text{otherwise} \end{cases}$$

It can be easily seen that ψ is complete and it verifies NNR, A and CS. Moreover, since it does not verify condition (ii) of theorem 4.2 we can conclude that it is irrational. In this case, chains with six alternatives may lead to irrationality. However, any chain with three alternatives cannot lead to irrationality. Indeed, given a chain $G = (x - y - z - x)$ two cases are possible

(a) there exist individuals i, j, l such that

$$\mu^i(x,y) + \mu^i(y,x) > 1$$
$$\mu^j(x,z) + \mu^j(z,x) > 1$$
$$\mu^l(y,z) + \mu^l(z,y) > 1 \ .$$

Then the cycle of indifferences $xIyIzIx$ will have positive weight for the aggregated relation μ. This is due to the fact that if at least one individual has positive indifference over a pair of alternatives, then the aggregated preference will have positive indifference over the same pair. For example, since $\mu^i(x,y) + \mu^i(y,x) > 1$ then

$$\mu(x,y) + \mu(y,x) \geq \frac{\sum_{j'=1}^n (\mu^{j'}(x,y) + \mu^{j'}(y,x))}{n} > 1$$

because all relations are complete.

Example 4.3 Another example of intensity aggregation rule which can easily be proven to be NAI by applying theorem 4.1 to prove completeness and theorem 4.2 to prove non absolute irrationality, is the following

$$\gamma(a^1, \ldots, a^n) = \frac{\min\{a^1, \ldots, a^n\} + \max\{a^1, \ldots, a^n\}}{2}$$

□

Notice that all of the intensity aggregation rules given in the examples above, verify NNR, A, U, CS and ND.

5 The importance of the number of alternatives

It is clear that one of the key points of our approach is that we suppose no knowledge about the number of alternatives. We assumed that NAI intensity aggregation rules should never lead to absolute irrationality, independently from the number of alternatives.

In our context, of course, there is no decision problem when there is only one feasible alternative x, and we expect absolute rationality. Indeed, the values $\mu^i(x,x) = 1$ for all i could have been easily assumed by hypothesys and then the aggregated value, under general ethical conditions, would have been $\mu(x,x) = 1$.

If there are just two feasible alternatives we could say that in fact there are no irrational rules. Notice that the preference values $\mu(x,y)$ and $\mu(y,x)$ will never lead by themselves to absolute irrationality, because in view of equation 2.4, $A_\mu(\{x - y - x\}) = 0$ if and only if

$$\mu_B(x,y)^2 + \mu_I(x,y)^2 + \mu_W(x,y)^2 = 0$$

which is impossible since $\mu_B(x,y) + \mu_I(x,y) + \mu_W(x,y) = 1$.

Therefore, any intensity aggregation rule will produce aggregated preferences μ such that $A_\mu(G) > 0$ for all chains G with two elements. This is consistent with Arrow's crisp model which does not lead to an Impossibility Theorem when the number of alternatives is either one or two. In general, ethical rules with absolute rationality are not possible in the crisp context when three or more alternatives are present, but they are possible if the absolute rationality condition is dropped out.

$0 < \alpha \leq \frac{1}{2}$, consider the following example with two individuals and three alternatives x, y, z

$$\begin{aligned}
\mu^i(x,y) = \mu^i(y,x) &= 1 \ \ \forall i = 1,2 \\
\mu^i(y,z) = \mu^i(z,y) &= 1 \ \ \forall i = 1,2 \\
\mu^1(z,x) = \mu^2(x,z) &= 1 \\
\mu^1(x,z) = \mu^2(z,x) &= \alpha \ .
\end{aligned}$$

In this case, the two individuals are non irrational since $\alpha > 0$. However, the aggregation rule μ verifies

$$\begin{aligned}
\mu(x,y) = \mu(y,x) &= 1 \\
\mu(y,z) = \mu(z,y) &= 1 \\
\mu(x,z) = \mu(z,x) &= \alpha \ .
\end{aligned}$$

Therefore, μ is irrational for $\alpha = \frac{1}{2}$ and non complete for $\alpha < \frac{1}{2}$.

Let us then fix $\frac{1}{2} < \alpha < 1$. It is clear that condition (ii) is verified. Let us then prove that condition (i) also holds.

We want to prove that $\alpha(a^1, \ldots, a^n) + \alpha(b^1, \ldots, b^n) > 1$ if for all $i = 1, 2, \ldots, n$, $a^i + b^i > 1$. Several cases are possible and they can all be trivially reduced to one of the following

1. $\alpha(a^1, \ldots, a^n) \geq \alpha$ and $\alpha(b^1, \ldots, b^n) \geq \alpha$: the result is true since $\alpha > \frac{1}{2}$.

2. $\alpha(a^1, \ldots, a^n) = \sup_i a^i$ and $\alpha(b^1, \ldots, b^n) = \alpha$: in this case there exists j such that $b^j \leq \alpha$. Therefore, $\sup_i a^i + \alpha \geq \sup_i a^i + b^j \geq a^j + b^j > 1$.

3. $\alpha(a^1, \ldots, a^n) = \sup_i a^i$ and $\alpha(b^1, \ldots, b^n) = \sup_i b^i$: we have $\sup_i a^i + \sup_i b^i \geq a^1 + b^1 > 1$.

4. $\alpha(a^1, \ldots, a^n) = \inf_i a^i$ and $\alpha(b^1, \ldots, b^n) = \sup_i b^i$: then, if $a^j = \inf_i a^i$ we have

$$a^j + \sup_i b^i \geq a^j + b^j > 1.$$

Completeness can be proven in an analogous manner. Therefore, in view of remark 4.1 we can conclude that the rule is NAI for any $\frac{1}{2} < \alpha < 1$.

□

Proof. Trivial, since NNR and CS imply that $\phi(1,\ldots,1) = 1$ and $\phi(0,\ldots,0) = 0$.

COROLLARY 4.2 *Let $\phi : [0,1]^n \to [0,1]$ be a complete intensity aggregation rule verifying condition PR. Then ϕ is NAI.*

Proof. In view of remark 4.1 we only need to prove that (i) and (ii) are verified.

To prove that (i) is verified, observe that from the completeness hypothesis and from theorem 4.1 we have

$$\phi(a^1,\ldots,a^n) + \phi(b^1,\ldots,b^n) > \phi(a^1,\ldots,a^n) + \phi(1-a^1,\ldots,1-a^n) \geq 1$$

whenever $a^i + b^i > 1$ for all $i = 1,2,\ldots,n$.

(ii) follows immediately from PR. ■

Example 4.1 By applying Corollary 4.2 we can easily prove that the following intensity aggregation rule (see [4], pg. 60) is NAI:

- *Weighted Generalized Mean:* given $w_1,\ldots,w_n$, positive real numbers and $r \geq 1$, then

$$WGM(a^1,\ldots,a^n) = \left(\sum_{i=1}^{n} \frac{w^i (a^i)^r}{\sum_{j=1}^{n} w^j}\right)^{\frac{1}{r}}$$

  where the assumption $r \geq 1$ has been made in order to assure completeness.

□

Example 4.2 By applying theorem 4.2 we can prove that some elements of the class of mixed rules defined in [3] are NAI. Given $\alpha \in [0,1]$ the corresponding rule is defined as

$$\alpha(a^1,\ldots,a^n) = \begin{cases} \inf_i a^i & \text{if } a^i \geq \alpha \text{ for all } i = 1,2,\ldots,n \\ \sup_i a^i & \text{if } a^i \leq \alpha \text{ for all } i = 1,2,\ldots,n \\ \alpha & \text{otherwise} \end{cases} .$$

If $\alpha = 1$ we obtain the maximum rule (cfr. 4.9) and therefore we have a complete but irrational rule. If $\alpha = 0$ we obtain the minimum rule (cfr. 4.8) and therefore we have an incomplete rule. If

Notice that $\mu_I^i(x_{n+1}, x_1) = 0$, therefore for each individual i we have:

$$\begin{aligned}
\Pi_{k=1}^{n+1}\mu^i(x_k, x_{k+1}) &= \Pi_{k=1}^{n} a^{R(i+k-1)} \\
\Pi_{k=1}^{n+1}\mu^i(x_{k+1}, x_k) &= 0 \\
\Pi_{k=1}^{n+1}\mu_I^i(x_k, x_{k+1}) &= 0
\end{aligned}$$

where $x_{n+2} = x_1$. Since $a^j < 1$ for some j, $\Pi_{k=1}^{n} a^{R(i+k-1)} \neq 1$ which implies that μ^i is non irrational.

For the aggregated opinion μ we have that for all $k = 1, 2, \ldots, n$,

$$\mu(x_k, x_{k+1}) = \phi(a^j, a^{j+1}, \ldots, a^n, a^1, \ldots, a^{j-1}).$$

Therefore, because of the anonimity condition for all $k = 1, \ldots, n$

$$\mu(x_k, x_{k+1}) = \phi(a^1, \ldots, a^n) = 1.$$

Moreover, from the hypotheses we have

$$\mu(x_1, x_{n+1}) = 0 \ , \ \mu(x_{n+1}, x_1) = 1$$

which imply that $\mu_I(x_{n+1}, x_1) = 0$. So,

$$\begin{aligned}
\Pi_{k=1}^{n+1}\mu(x_k, x_{k+1}) &= 1 \\
\Pi_{k=1}^{n+1}\mu(x_{k+1}, x_k) &= 0 \\
\Pi_{k=1}^{n+1}\mu_I(x_k, x_{k+1}) &= 0
\end{aligned}$$

which in turn imply that μ is irrational, contradicting the initial hypothesys.

The theorem is then proven. ■

Remark 4.1 From the first part of the proof of theorem 4.2 we can conclude that in order for a complete intensity aggregation rule to be NAI, conditions (i) and (ii) are sufficient. □

We now have two immediate corollaries of theorem 4.2.

COROLLARY 4.1 *A complete intensity aggregation rule $\phi : [0,1]^n \to [0,1]$ verifying conditions NNR, CS and A is NAI if and only if conditions (i)-(ii) of theorem 4.2 hold.* ■

(case 1) (i) is not verified. Therefore,

$$\phi(a^1,\ldots,a^n)+\phi(b^1,\ldots,b^n)=1$$

for some $(a^1,\ldots,a^n),(b^1,\ldots,b^n)\in[0,1]^n$ such that $a^i+b^i>1$ for all $i=1,2,\ldots,n$. If we then define for all i

$$\mu^i(x,y)=\mu^i(y,x)=1\ ,\ \mu^i(y,z)=\mu^i(z,y)=1\ ,\ \mu^i(z,x)=a^i\ ,\ \mu^i(x,z)=b^i$$

for three different alternatives x,y,z, and put $G=(x-y-z-x)$ we have that for all $i=1,2,\ldots,n$,

$$\begin{aligned}\mu^i(x,y)\mu^i(y,z)\mu^i(z,x) &= a^i\\ \mu^i(y,x)\mu^i(z,y)\mu^i(x,z) &= b^i\\ \mu^i_I(x,y)\mu^i_I(y,z)\mu^i_I(z,x) &= \mu^i_I(x,z)=a^i+b^i-1\ .\end{aligned}$$

So, since $1-(a^i+b^i-2(a^i+b^i-1))=1-(2-(a^i+b^i))=a^i+b^i-1>0$ for all $i=1,2,\ldots,n$, we have that each μ^i is non irrational. However, since $\phi(1,\ldots,1)=1$, for the aggregated preference μ we have

$$\mu(x,y)=\mu(y,x)=\mu(y,z)=\mu(z,y)=1\ ,\ \mu_I(x,z)=0$$

and so

$$\mu(x,y)\mu(y,z)\mu(z,x)+\mu(y,x)\mu(z,y)\mu(x,z)-2\mu_I(x,y)\mu(y,z)\mu_I(z,x)=\mu(z,x)+\mu(x,z)=1$$

which implies that $A(\mu)=0$. So, μ is irrational, which contradicts the initial hypothesis.

(case 2) (ii) is not verified. Therefore, there exists $(a^1,\ldots,a^n)\in[0,1]^n$ such that $\phi(a^1,\ldots,a^n)=1$ with $a^j<1$ for some j. Let then $G=(x_1-x_2-\cdots-x_{n+1}-x_1)$ be a chain with $n+1$ alternatives. Moreover, for each individual $i=1,2,\ldots,n$, define

$$\mu^i(x_{n+1},x_1)=1\ ,\ \mu^i(x_1,x_{n+1})=0\ ,\ \text{ and }\mu^i(x_k,x_{k+1})=a^{R(i+k-1)}\ ,\ \mu^i(x_{k+1},x_k)=1,$$

for each $k=1,\ldots,n$, where $R(j)=j \bmod n$.

Proof. Suppose first that ϕ is complete and verifies conditions (i)-(ii). Given an arbitrary chain of alternatives $G = (x_1 - x_2 - \cdots - x_m - x_1)$ and denoted by μ the aggregated preference corresponding to ϕ, in view of the rationality of the individuals two cases are possible:

(a) there exists an individual rational cycle with some strict preference, i.e. there exists i and a rational cycle $x_1 P_1 x_2 P_2 \cdots x_m P_m x_1$ such that

- there exists h such that $P_h = B$ and then since the cycle is rational there exists $k \neq h$ such that $P_k = W$;
- $\Pi_{h'=1}^{m} \mu^i_{P_{h'}}(x_{h'}, x_{h'+1}) > 0$.

Notice that $\mu^i_B(x_h, x_{h+1}) > 0$ implies $\mu^i(x_{h+1}, x_h) < 1$. Hence, in view of (ii) we have $\mu(x_{h+1}, x_h) < 1$ and then $\mu_B(x_h, x_{h+1}) > 0$. Analogously, from $\mu^i_W(x_k, x_{k+1}) > 0$ we can deduce that $\mu_W(x_k, x_{k+1}) > 0$. Let us then consider the cycle

$$x_1 P_1 x_2 P_2 \cdots x_{h-1} P_{h-1} x_h B x_{h+1} \cdots x_k W x_{k+1} \cdots x_m P_m x_1$$

This cycle will be rational no matter how we define $P_{h'}$ in $\{W, I, B\}$ for any h' different from h and k. Moreover, since for any pair $(x_{h'}, x_{h'+1})$

$$\mu_B(x_{h'}, x_{h'+1}) + \mu_I(x_{h'}, x_{h'+1}) + \mu_W(x_{h'}, x_{h'+1}) = 1 \ ,$$

it must be the case that at least one of the three values above is greater than zero. So, by choosing one of such positive values for each h', we can certainly build a rational cycle for G whose weight is greater than zero. Therefore, $A_\mu(G) > 0$.

(b) For any individual $i = 1, 2, \ldots, n$, the rational cycle of all individual indifferences, $x_1 I x_2 I \cdots I x_m I x_1$, has positive weight. Thus, we have $\mu^i_I(x_h, x_{h+1}) > 0$ for all $i = 1, 2, \ldots, n$ and for all $h = 1, 2, \ldots, m$, which implies that $\mu^i(x_h, x_{h+1}) + \mu^i(x_{h+1}, x_h) > 1$. Therefore, in view of condition (i) we have $\mu(x_h, x_{h+1}) + \mu(x_{h+1}, x_h) > 1$ which in turn implies that $\mu_I(x_h, x_{h+1}) > 0$ for all $h = 1, 2, \ldots, m$. Once again, $A_\mu(G) > 0$.

Since G was an arbitray chain, from case (a) and (b) above we can conclude that μ and in turn ϕ are not irrational whenever conditions (i)-(ii) are satisfied.

Conversely, suppose that ϕ verifies the hypotheses of the theorem. Reasoning by contradiction, two cases are possible:

may lead to irrational social preference relations (see theorem 4.2 below), even though it assures completeness.

As a direct consequence of the above definitions we have

LEMMA 4.1 *Given an intensity aggregation rule* $\phi : [0,1]^n \to [0,1]$ *the associated aggregated fuzzy preference relation* μ *is complete, for any profile of complete individual preferences, if and only if*

$$\phi(a^1, \ldots, a^n) + \phi(b^1, \ldots, b^n) \geq 1$$

whenever $a^i + b^i \geq 1$ *for all* $i = 1, 2, \ldots, n$. ■

In view of lemma 4.1, we say that an aggregation rule ϕ is complete if and only if $\phi(a^1, \ldots, a^n) + \phi(b^1, \ldots, b^n) \geq 1$ whenever $a^i + b^i \geq 1$ for all $i = 1, 2, \ldots, n$.

A useful characterization of complete intensity aggregations is given by the following theorem.

THEOREM 4.1 *An intensity aggregation rule* $\phi : [0,1]^n \to [0,1]$ *verifying NNR is complete if and only if*

$$\phi(a^1, \ldots, a^n) + \phi(1 - a^1, \ldots, 1 - a^n) \geq 1$$

for all $(a^1, \ldots, a^n) \in [0,1]^n$.

Proof. If $a^i + b^i \geq 1$ for all $i = 1, 2, \ldots, n$, then in view of NNR it must be

$$\phi(a^1, \ldots, a^n) + \phi(b^1, \ldots, b^n) \geq \phi(a^1, \ldots, a^n) + \phi(1 - a^1, \ldots, 1 - a^n) \geq 1.$$

The converse is trivial. ■

The following results allow us to characterize our non irrational aggregation rules.

THEOREM 4.2 *Let* $\phi : [0,1]^n \to [0,1]$ *be a complete intensity aggregation rule verifying condition A and such that* $\phi(1, \ldots, 1) = 1$ *and* $\phi(0, \ldots, 0) = 0$. *Then* ϕ *is NAI if and only if the following conditions hold:*

(i) if $a^i + b^i > 1$ *for all* $i = 1, 2, \ldots, n$; *then* $\phi(a^1, \ldots, a^n) + \phi(b^1, \ldots, b^n) > 1$;

(ii) $\phi(a^1, \ldots, a^n) = 1$ *implies* $a^i = 1$ *for all* $i = 1, 2, \ldots, n$.

(ND) *Non Dictatorship*: there is no individual i such that

$$\phi(a^1, a^2, \ldots, a^n) = a^i$$

for any $(a^1, \ldots, a^{i-1}, a^{i+1}, \ldots, a^n) \in [0,1]^{n-1}$.

It is easy to see that PR implies NNR, A implies ND and U implies CS. NNR (or PR) and CS together imply that $\phi(1, \ldots, 1) = 1$ and $\phi(0, \ldots, 0) = 0$. At any rate, conditions NNR, CS and A can be considered a must in a general context and they are not in fact inconsistent. Many aggregation rules verifying all of the above conditions (or a meaningful subset of them) can be defined. For instance, the mean rule defined in 2.7 satisfies all the of the above conditions.

4 Non irrational intensity aggregation rules

Following [9], absolute rationality cannot be claimed in many practical situations even for individuals. So, we should look for some non absolute irrationality results. Notice that any possibility result in this context will justify the search for aggregation rules that are in some sense as good as possible. Therefore, we ask whether or not there exist aggregation rules assuring non absolutely irrational aggregated preferences, according to the following definition.

DEFINITION 4.1 *Given n individuals, an intensity aggregation rule $\phi : [0,1]^n \to [0,1]$ is* non absolutely irrational *(NAI), or simply* non irrational, *if for any arbitrary finite set of alternatives X, the associated aggregated preference $\mu : X \times X \to [0,1]$ is complete and non absolutely irrational, i.e. $A(\mu) > 0$, whenever all individuals are complete and non absolutely irrational themselves, i.e. $A(\mu^i) > 0$ for all $i = 1, 2, \ldots, n$, with $\mu^i : X \times X \to [0,1]$ for all i.* □

Notice that in this way both individual and social opinions are required to belong to the set of Non-Absolutely Irrational (NAI) complete fuzzy preference relations. Therefore, we are in fact modifying the Unrestricted Domain condition.

For example, the *minimum* rule

$$m(a^1, \ldots, a^n) = \min_{i=1,\ldots,n} a^i \tag{4.8}$$

cannot be considered because it does not guarantee completeness, and the *maximum* rule

$$M(a^1, \ldots, a^n) = \max_{i=1,\ldots,n} a^i \tag{4.9}$$

It is clear that if we also assume a standard neutrality condition in order to assure that the same intensity aggregation mapping ϕ will be associated to any pair of alternatives, each possible aggregation procedure is characterized by one of these intensity aggregation mappings. Such a neutrality condition is

(N) *Neutrality*: given any permutation of the set of alternatives π, if $\nu^i(x,y) = \mu^i(\pi(x),\pi(y))$ for all $i = 1,2,\ldots,n$ and any pair of alternatives x, y, then

$$\phi(\nu^1(x,y),\cdots,\nu^n(x,y)) = \phi(\mu^1(\pi(x),\pi(y)),\cdots,\mu^n(\pi(x),\pi(y)))$$

For the time being, we will suppose that conditions IIA, UD, N hold.

Some *ethical* conditions may also be imposed to the intensity aggregation rules:

(NNR) *Non Negative Responsiveness*:

$$\phi(a^1,a^2,\ldots,a^n) \geq \phi(b^1,b^2,\ldots,b^n)$$

if $a^i \geq b^i$ for all $i = 1,2,\ldots,n$.

(PR) *Positive Responsiveness*:

$$\phi(a^1,a^2,\ldots,a^n) > \phi(b^1,b^2,\ldots,b^n)$$

if $a^i \geq b^i$ for all $i = 1,2,\ldots,n$ and there exists $1 \leq j \leq n$ such that $a^j > b^j$.

(A) *Anonimity*: given any permutation of the set of individuals $\pi : \{1,\ldots,n\} \to \{1,\ldots,n\}$, we have

$$\phi(a^1,a^2,\ldots,a^n) = \phi(a^{\pi(1)},\ldots,a^{\pi(n)}).$$

(U) *Unanimity*: if $a^i = a$ for all $i = 1,2,\ldots,n$, then

$$\phi(a^1,a^2,\ldots,a^n) = a.$$

(CS) *Citizen Sovereign*: for any given $a \in [0,1]$ there exists a profile $(a^1,a^2,\ldots,a^n) \in [0,1]^n$ such that

$$\phi(a^1,a^2,\ldots,a^n) = a.$$

In the following sections we will deal with the amalgamation of complete fuzzy preferences μ which are non absolutely irrational. As it has been proven in [8], if the set of alternatives X is fixed then there exist aggregation rules that assure non irrational social preferences for all possible profiles of n non absolutely irrational individual opinions. That is the case, for example, of the mean rule, defined for $x, y \in X$ as

$$\sigma(\mu^1, \ldots, \mu^n)(x, y) = \frac{\sum_{i=1}^{n} \mu^i(x, y)}{n} \tag{2.7}$$

As pointed out previously, we will not suppose that the set of alternatives is fixed and a necessary and sufficient condition for such *non irrational* rules is given, generalizing in this way a partial result given in [9].

3 Intensity amalgamation rules and ethical conditions

Once a group of $n \geq 2$ individuals is fixed, we should be able to aggregate their opinions about any set of alternatives in a coherent way. Therefore, we have to define aggregation operations that can take into account any extra alternative x so to properly extend any previous aggregated opinion relative to a collection of alternatives not containing x. Our proposal is based upon amalgamation mappings which will allow us the successive amalgamation of alternatives preferences one by one, and the key properties are the standard conditions

(IIA) *Independence of Irrelevant Alternatives*: each aggregated preference relation $\mu(x, y)$ depends solely on the values $\mu^i(x, y)$, i.e. on the individual preference intensities of x over y.

(UD) *Unrestricted Domain*: the aggregation rule is defined over all possible profiles of fuzzy preferences.

Under these conditions, we propose the following definition

DEFINITION 3.1 *An intensity aggregation rule is any mapping $\phi : [0, 1]^n \to [0, 1]$ which assigns a fuzzy preference intensity to each profile of individuals fuzzy preference intensities.* □

Therefore, given ϕ and any profile $(\mu^1(x, y), \cdots, \mu^n(x, y)) \in [0, 1]^n$ we can define the aggregated value

$$\mu(x, y) = \phi(\mu^1(x, y), \cdots, \mu^n(x, y)).$$

which implies that 2.5 holds.

Fixed a finite set of alternatives X, in view of 2.5, rationality can be defined as a fuzzy property $A : \mathcal{P}(X) \to [0,1]$ with

$$A(\mu) = \min_G A_\mu(G) \tag{2.6}$$

and where $\mathcal{P}(X)$ is the set of all complete fuzzy preferences. The degree of rationality is then associated to the minimum degree of acyclicity along all chains (see [6, 7]).

Notice that complete fuzzy preference relations can be absolutely rational (i.e. $A(\mu) = 1$) or absolutely irrational (i.e. $A(\mu) = 0$) without being a crisp preference relation, as we will show in the next two examples. Obviously, a crisp complete order will be absolutely rational if it is a linear order and absolutely irrational otherwise.

Example 2.1 Let us consider the following fuzzy preference relation defined on the set of alternatives $\{x, y, z\}$ as:

$$\mu(x,y) = \mu(y,x) = \mu(y,z) = \mu(z,y) = 1 \ , \ \mu(x,z) = \mu(z,x) = \frac{1}{2}.$$

Such a fuzzy preference is absolutely irrational, that is $A(\mu) = 0$. Indeed,

$$\begin{aligned} \mu(x,y)\mu(y,z)\mu(z,x) &= \frac{1}{2} \\ \mu(y,x)\mu(z,y)\mu(x,z) &= \frac{1}{2} \\ \mu_I(x,y)\mu_I(y,z)\mu_I(z,x) &= 0 \ . \end{aligned}$$

Therefore, from 2.5 we have $A(\mu) = 0$. □

Example 2.2 Let us consider now the fuzzy preference μ defined on the set of alternatives $\{x, y, z\}$ as:

$$\mu(x,y) = \mu(z,y) = 1 \ , \ \mu(y,x) = \mu(y,z) = 0 \ , \ \mu(x,z) = \mu(z,x) = \frac{1}{2}.$$

Then,

$$\begin{aligned} \mu(x,y)\mu(y,z)\mu(z,x) &= 0 \\ \mu(y,x)\mu(z,y)\mu(x,z) &= 0 \\ \mu_I(x,y)\mu_I(y,z)\mu_I(z,x) &= 0 \end{aligned}$$

and, from 2.5 we have $A(\mu) = 1$. □

$$\mu_B(x,y)\mu_W(y,z)\mu_I(z,x) + \mu_B(x,y)\mu_W(y,z)\mu_W(z,x) +$$
$$\mu_I(x,y)\mu_B(y,z)\mu_W(z,x) + \mu_I(x,y)\mu_I(y,z)\mu_I(z,x) + \mu_I(x,y)\mu_W(y,z)\mu_B(z,x) +$$
$$\mu_W(x,y)\mu_B(y,z)\mu_B(z,x) + \mu_W(x,y)\mu_B(y,z)\mu_I(z,x) +$$
$$\mu_W(x,y)\mu_B(y,z)\mu_W(z,x) + \mu_W(x,y)\mu_I(y,z)\mu_B(z,x) + \mu_W(x,y)\mu_W(y,z)\mu_B(z,x) \ .$$

In general, given a fuzzy preference relation μ and given a cycle $C \equiv x_1 P_1 \cdots x_k P_k x_1$ where $P_h \in \{B, I, W\}$ for all $h = 1, 2, \ldots, k$, the natural weight associated to C and denoted by $\Delta(C)$ will be

$$\Delta(C) = \Pi_{h=1}^{k} \mu_{P_h}(x_h, x_{h+1})$$

where $x_{k+1} = x_1$ for convenience.

Therefore, given a chain $G = (x_1 - x_2 - \cdots - x_k - x_1)$ a natural degree of rationality associated to G and denoted by $A_\mu(G)$ can be defined as

$$A_\mu(G) = \sum_{C \in \text{ rat.cycles}} \Delta(C).$$

We will now prove that $A_\mu(G)$ verifies

$$A_\mu(G) = 1 - (\Pi_{h=1}^{k} \mu(x_h, x_{h+1}) + \Pi_{h=1}^{k} \mu(x_{h+1}, x_h) - 2\Pi_{h=1}^{k} \mu_I(x_h, x_{h+1})). \tag{2.5}$$

Because of 2.3 we have $\Pi_{h=1}^{k}(\mu_B(x_h, x_{h+1}) + \mu_I(x_h, x_{h+1}) + \mu_W(x_h, x_{h+1})) = 1$, thus

$$\sum_{\text{rat.cycles}} \Delta(C) = 1 - \sum_{\text{irrat.cycles}} \Delta(C).$$

On the other hand,

$$\Pi_{h=1}^{k} \mu(x_h, x_{h+1}) + \Pi_{h=1}^{k} \mu(x_{h+1}, x_h) - 2\Pi_{h=1}^{k} \mu_I(x_h, x_{h+1}) =$$
$$\Pi_{h=1}^{k}(\mu_B(x_h, x_{h+1}) + \mu_I(x_h, x_{h+1})) + \Pi_{h=1}^{k}(\mu_W(x_h, x_{h+1}) + \mu_I(x_h, x_{h+1})) - 2\Pi_{h=1}^{k} \mu_I(x_h, x_{h+1}) =$$
$$\sum_{C \in \mathcal{C}_B} \Delta(C) + \sum_{C' \in \mathcal{C}_W} \Delta(C') - 2\Pi_{h=1}^{k} \mu_I(x_h, x_{h+1}) \ ,$$

where $\mathcal{C}_B$ is the collection of all cycles of type $x_1 P_1 x_2 P_2 \cdots x_k P_k x_1$ with $P_h \in \{B, I\}$ for all $h = 1, 2, \ldots, k$, and $\mathcal{C}_W$ is the collection of all cycles of type $x_1 P_1' x_2 P_2' \cdots x_k P_k' x_1$ with $P_h' \in \{W, I\}$, for all h. So, $\mathcal{C}_B \cup \mathcal{C}_W$ is the collection of all the irrational cycles plus the cycle of all indifferences, which in particular belongs to both $\mathcal{C}_B$ and $\mathcal{C}_W$. Thus,

$$\sum_{C \in \mathcal{C}_B} \Delta(C) + \sum_{C' \in \mathcal{C}_W} \Delta(C') - 2\Pi_{h=1}^{k} \mu_I(x_h, x_{h+1}) = \sum_{C \in \text{ irrat.cycles}} \Delta(C)$$

preference of y over x (xWy, x *is worse than* y). Hence, fuzzy preferences are modeled according to a fuzzy partition with three classes, in such a way that

$$\mu_B(x,y)+\mu_I(x,y)+\mu_W(x,y)=1 \ \ \forall x,y \tag{2.3}$$

A cycle of preferences will be defined over chains $G=(x_1-x_2-\cdots-x_k-x_1)$ of k distinct alternatives as

$$x_1P_1x_2P_2\cdots x_kP_kx_1$$

where $P_h\in\{W,I,B\}$ for all $h=1,2,\ldots,k$. A cycle $x_1P_1x_2P_2\cdots x_kP_kx_1$ is *irrational* if either

- $P_h\in\{B,I\}$ for all $h=1,2,\ldots,k$ and $B\in\{P_h:h=1,2,\ldots,k\}$; or
- $P_h\in\{W,I\}$ for all $h=1,2,\ldots,k$ and $W\in\{P_h:h=1,2,\ldots,k\}$.

We say that a cycle is rational if it is not irrational. Then, given any fuzzy preference μ over a fixed set of alternatives and a chain of alternatives, we can look for all possible rational cycles of preferences, weigh them in a natural way and assign to the chain a degree of rationality.

For example, given a chain $(x-x)$ with only one alternative, we obtain two distinct cycles, xBx and xIx, of which only xIx is a rational cycle. The degree of *rationality* of such a chain $(x-x)$ will be just $\mu_I(x,x)$. If we consider the chain $(x-y-x)$ with two alternatives, only the cycles $xByWx, xWyBx$ and $xIyIx$, are rational. So, we can propose

$$\mu_B(x,y)\mu_W(y,x)+\mu_W(x,y)\mu_B(y,x)+\mu_I(x,y)\mu_I(x,y)=\mu_B(x,y)^2+\mu_W(x,y)^2+\mu_I(x,y)^2 \tag{2.4}$$

as the natural degree of rationality for such a 2-element chain. This procedure can be extended to chains containing three or more alternatives, by adding along all possible rational cycles the product of the intensities associated to each preference in such a cycle. For example, the chain $(x-y-z-x)$ with three alternatives gives the following thirteen rational cycles

$$\begin{array}{c} xByBzWx \ \ xByIzWx \ \ xByWzBx \ \ xByWzIx \ \ xByWzWx \\ xIyBzWx \ \ xIyIzIx \ \ xIyWzBx \\ xWyBzBx \ \ xWyBzIx \ \ xWyBzWx \ \ xWyIzBx \ \ xWyWzBx \end{array}$$

and fourteen irrational cycles. So, the degree of rationality associated to such a chain will be

$$\mu_B(x,y)\mu_B(y,z)\mu_W(z,x)+\mu_B(x,y)\mu_I(y,z)\mu_W(z,x)+\mu_B(x,y)\mu_W(y,z)\mu_B(z,x)+$$

comparability between pairs of alternatives. More recently in [2], completeness has been proposed as a measure of support of the set of alternatives. Here, we will follow this second idea and we will assume completeness with the intended meaning that all individuals consider the set of alternatives, on which they are expressing their opinion, feasible and comprehensive. In this paper, the amalgamation of preferences will not be obtained according to Arrow's model (rules for a fixed set of alternatives) but through intensity aggregation rules which will allow the successive aggregation of alternatives. This approach seems to be in principle more realistic, since for instance most committees begin to analyze the given problem without an a priori fixed number of final alternatives to be voted.

To clarify the above comment consider a faculty search committee. First, apart from the number of applications, they set some rules that will guide the decision process. Then, once the deadline for the applications expires, they start reviewing the candidates whose CV's have already arrived. Some candidates are discarded and some remain and will be compared to new ones, whose CV's will be received later on. Starting the faculty search as early as possible allows the speeding up of the decision process, therefore aggregation rules that are independent from the number of alternatives are clearly needed.

2 Rationality as a fuzzy property

Let $\mu : X \times X \to [0,1]$ be a fuzzy preference relation over an arbitrary finite set of alternatives X. $\mu(x,y)$ represents the degree to which the relation x *not worse than* y holds. Let us assume that μ is complete in the sense that

$$\mu(x,y) + \mu(y,x) \geq 1 \quad \forall x, y \in X \tag{2.1}$$

Then, the values

$$\begin{aligned} \mu_I(x,y) &= \mu(x,y) + \mu(y,x) - 1 \\ \mu_B(x,y) &= \mu(x,y) - \mu_I(x,y) \\ \mu_W(x,y) &= \mu(y,x) - \mu_I(x,y) \end{aligned} \tag{2.2}$$

can be understood, respectively, as the degree to which the two alternatives are indifferent (xIy), the degree of strict preference of x over y, (xBy, x *is better than* y) and the degree of strict

1 Introduction

Arrow's paradox (cfr. [1]) in group decision making has been translated in the past into a fuzzy context and it has been shown that such a negative result can be avoided in several ways.

The original problem was formalized by Arrow in the following way: *given two or more individuals and supposing that each individual defines a (crisp) complete linear order over a fixed set of alternatives, can we always define in an ethical way a social (crisp) complete linear order corresponding to the given profile of social opinions ?*

Therefore, it was assumed that each individual was able to define in a consistent way which alternative was the best among any possible pair of alternatives, and so for the group itself. Two basic approaches can be found in fuzzy literature, depending on how intensities of preferences are introduced. In [3] for example, it is assumed that individuals and group opinions are given in terms of fuzzy sets of feasible alternatives, so that each individual i defines a degree $\mu_i(x)$ of feasibility for each alternative x. These feasibility values have to be aggregated into social degrees of feasibility $\mu(x)$. Morevoer, the set of individuals is not a priori fixed and social values are obtained through a successive aggregation of individual opinions, in a one by one fashion. The main result in [3] shows that only a class of mixed rules were possible in that context (see example 4.2). As it was proven in [5], the reason for such a restrictive result is based on the fact that the model does not take into account the number of individuals supporting each alternative, which does not seem to be *ethical.*

More coherently with Arrow's welfare approach is to assume fuzzy binary preference relations. In this case, individual degrees of preference $\mu_i(x, y)$ of the alternative x over the alternative y are defined for each pair of alternatives. These preferences have then to be aggregated into social preferences $\mu(x, y)$. As shown in [10], just the assumption of social fuzzy preferences is enough to avoid Arrow's paradox (i.e. collections of reasonable ethical conditions are not inconsistent in this context). But in [6, 7], it was pointed out that the ethical conditions were not the key point in Arrow's problem, as much as the underlying idea of rationality (only complete linear orders were assumed). Obviously, some ethical conditions must also be assumed in the fuzzy context as well as some degree of rationality for the preference relations.

In this paper we will assume non absolutely irrational (in the sense of [6, 7]) complete fuzzy preference relations. The hypothesis of completeness has traditionally been assumed to formalize

A characterization of rational amalgamation operations

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Abstract

This paper deals with amalgamation of fuzzy opinions when a fixed number of individuals is faced with an unknown number of alternatives. The aggregation rule is defined by means of intensity agggregation operations that verify certain ethical conditions, and assuming fuzzy rationality as defined in [6, 7]. A necessary and sufficient condition for non-irrationality is presented, along with comments on the importance of the number of alternatives.

Key words: Aggregation rules, fuzzy preferences, group decision making.

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