Analysis of Random Processes via Or-And Tree Evaluation

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1 Introduction

We introduce a new set of probabilistic analysis tools related to the amplification method introduced by [12] and further developed and used in [13], [5]. These tools provide a unifying, intuitive, and powerful framework for carrying out the analysis of several previously studied random processes of interest, including the random loss-resilient codes introduced in [9], the greedy algorithm for matchings in random graphs studied in [7], and the threshold for solving random k-SAT formula using the pure literal rule [4]. In addition, generalization of these problems not previously analyzed can now be analyzed in a straightforward manner. For example, we can analyze generalizations of the processes considered in [9]. As another example, we can analyze the behavior of the pure literal rule on random k-SAT formula chosen from distributions not handled by previous analyses.

Our main tool is a simple analysis of the probability an Or-And tree formula evaluates to 1. The simple version of this Or-And tree evaluation problem is the following. Let T_ℓ be a tree of depth 2ℓ with each leaf node labeled with either 0 or 1. (The root of the tree is at depth 0, and the leaves are at depth 2ℓ.) Each node at depth $0, 2, 4, \ldots, 2\ell - 2$ is labeled as an "OR" gate (and it evaluates to the "OR" of its children), and each node at depth $1, 3, 5, \ldots, 2\ell - 1$ is labeled as an "AND" gate (and it evaluates to the "AND" of its children). We say the

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tree is (d_{or},d_{and})-*regular* if each "OR" node has d_{or} children and each "AND" node has d_{and} children.

Our analysis is related to the study of amplification, initiated by Moore and Shannon [12], and continued in several works [13], [5]. Consider the probability the root of the tree evaluates to 0 when the value of each leaf is independently chosen to be 0 with probability p. Let us denote this probability as y_ℓ. One typical amplification question with respect to Or-And trees is whether or not there is a threshold phenomena, i.e., is there a critical value ϕ such that if $p > \phi$ then y_ℓ goes to 1 as ℓ goes to infinity and if $p < \phi$ then as y_ℓ goes to 0 as ℓ goes to infinity. Of primary interest in these studies is the rate of amplification, i.e., what is the rate at which y_ℓ goes to either 0 or 1 as a function of ℓ.

One work that uses exactly this type of analysis is the elegant randomized construction given in [13] of a polynomial size monotone boolean formula that computes the majority function. The basic idea behind the construction and proof exploits the fact that a $(2,2)$-regular OR-AND tree has a critical value of $\phi = (3-\sqrt{5})/2$, and that if $y_{\ell-1} = \phi + \epsilon$ then $y_\ell > \phi + c\epsilon$ for a constant $c > 1$. (The analogous rate of divergence is true if $y_{\ell-1} = \phi - \epsilon$.) In further work, [3] and [5] provide beautiful proofs that the construction size of [13] is optimal, this time using amplification analysis to prove a lower bound.

Our work has a similar spirit to the amplification work, but it differs in several ways. We generalize to allow the number of children of each node to vary in the following way. Let $(\alpha_0, \alpha_1, \ldots, \alpha_A)$ be a probability vector, i.e., $\alpha_i \geq 0$ for all $i \in \{0, \ldots, A\}$ and $\sum_{i=0}^{A} \alpha_i = 1$. Similarly, let $(\beta_0, \beta_1, \ldots, \beta_B)$ be a probability vector. Starting at the root and working down the tree, each "OR" node chooses to have i children with probability α_i independent of any other node, and similarly each "AND" node choose to have j children with probability β_j independent of any other node. (Some previous research introduced a variant of this form and used it in a limited way in their construction, e.g., [5].) A further generalization is to allow two positive values a and b such that each "OR" node is independently shortcircuited to produce the value 1 with probability a, and each "AND" node is independently shortcircuited to produce the value 0 with probability b.

Our new analysis is based on the following simple definitions and lemma. Define

$$\alpha(x) = \sum_{i=0}^{A} \alpha_i \cdot x^i, \tag{1}$$

$$\beta(x) = \sum_{i=0}^{B} \beta_i \cdot x^i, \tag{2}$$

$$f(x) = (1-a) \cdot \alpha(1 - (1-b) \cdot \beta(1-x)). \tag{3}$$

Lemma 1 *Define $y_0 = p$ to be the probability a leaf node is labeled 0. Then, for all $\ell \geq 1$, $y_\ell = f(y_{\ell-1})$.*

Although this lemma is simple to prove, as we shall see it is quite powerful.

The most significant difference between our work and previous work on amplification is that our goals are quite different: For example, for the loss-resilient codes, our final goal is to design $(\alpha_0, \ldots, \alpha_A)$ and $(\beta_0, \ldots, \beta_B)$ so that the average number of children per node is not too large, so that the ratio between the average number of children for an "OR" gate and for an "AND" gate satisfies a certain ratio, and so that for as small as possible value of a and as large as possible value of b the critical value of the tree is as close to 1 as possible.

Some of the analyses we present in the paper were previously done using different methodologies. For example, the analysis of the loss-resilient codes described in [9] was done by modeling the random process using differential equations, solving the equations to obtain a polynomial, and using a version of Kurtz's theorem [8] to make the connection between the behavior of the random process and that of the polynomial. (This type of approach was pioneered in the analysis of algorithms domain in [10].) One of the ingredients lacking in the previous analysis was a simple intuitive connection between the polynomial solution and the original process. With the new analysis, this intuitive connection is direct and compelling. In addition, the new analysis tools can be easily used to analyze important generalizations of the original process, which would have been much more difficult using the previous analysis. Finally, the new analysis is clearly the "right analysis", and will undoubtably find a number of other applications.

In the next three sections, we apply this simple lemma to the analysis of loss-resilient codes, the pure literal rule for random k-CNF formula, and the greedy matching algorithm for random graphs.

2 Loss-Resilient Code Analysis

2.1 Essentials of the Codes

The codes described in [9] consist of a cascading sequence of random bipartite graphs. To describe the encoding and decoding process, as well as to carry out the analysis, we need only consider one bipartite graph. Let G be a bipartite graph with n nodes on the left side, m nodes on the right side, and e edges in total between the nodes on the left and the right. We associate one message bit with each left node and one check bit with each right node. (This is for simplicity of description, in practice it is more efficient to associate several bytes of information with each node.) The encoding process computes the check bits from the message bits in the obvious way: the check bit associated with right node w is computed as the exclusive-or of all the message bits associated with the neighbors of w.

The entire encoding is transmitted, and we would like to recover all the message bits from a random fraction of the entire encoding, where this fraction is as small as possible. Assume inductively that all the check bits associated with the right nodes have already been recovered. Label the left nodes with a 0 if the associated message bit is missing, and with a 1 if the associated message bit has been either received directly or recovered indirectly as described below. The decoding process to recover the missing message bits invokes the following rule as long as it is applicable.

Recovery Rule: *The rule can be applied at any left node v with label 0 that has at least one right neighbor w such that all the left neighbors of w excluding v are labeled with a 1. The value of v can be recovered by computing the exclusive-or of the check bit associated with w and all the values associated with neighbors of w excluding v. Since the message bit associated with v has been recovered, the label of v is changed to 1 at this point.*

In terms of a graph process, the recovery rule can be written more succinctly as follows:

Graph Recovery Rule: *A left node v with label 0 is allowed to change its label to a 1 if it has at least one right neighbor w such that all left neighbors of w except v have label 1.*

The decoding process terminates successfully with all message bits recovered iff the graph recovery rule ends with no remaining left nodes with label 0.

2.2 New Analysis of the Original Process

The paper [9] gave an analysis of the decoding process described in the previous subsection using differential equations to model the process, and then solving these equations as polynomials. In this subsection, we obtain the same result using Lemma 1. The advantage of the analysis here is that it gives direct and intuitive insight into how the condition at the end arises. In the subsequent subsections we show how this new analysis can be used to derive several additional results.

Let $(p_0, p_1, \ldots, p_L)$ and $(q_0, q_1, \ldots, q_R)$ be probability vectors. As in [9], consider choosing a random bipartite graph with n left nodes and m right nodes as follows: Each node on the left is chosen to have degree i with probability p_i, and each node on the right is chosen to have degree j with probability p_j, where all choices are made independently. For there to be the same number of edges out of the left side as there out of the right side, we require that the number of edges e satisfies

$$e = n \cdot \sum_{i=0}^{L} i p_i = m \cdot \sum_{j=0}^{R} j q_j .$$

A random permutation π of $\{1, \ldots, e\}$ is chosen, and then, for all $i \in \{1, \ldots, e\}$, the edge with index i out of the left side is identified with the edge with index π_i out of the right side.

We are interested in properties of such a graph as n grows to infinity and with m fixed to cn for a fixed constant $c > 0$, and for fixed probability vectors

$(p_0, p_1, \ldots, p_L)$ and $(q_0, q_1, \ldots, q_R)$.

Consider the random subgraph G_ℓ of this graph obtained by the following process: Choose an edge (v, w) uniformly at random from among all edges, and then consider the subgraph G_ℓ induced by the left node v and all neighbors of v within distance 2ℓ after deleting the edge (v, w).

We claim that the probability G_ℓ fails to be a tree is proportional to $1/n$, i.e., asymptotically this probability goes to zero as n grows to infinity for a fixed value of ℓ. Furthermore, asymptotically the distribution on the shape of G_ℓ can be described as follows. For all $i = 1, \ldots, L$, let $\lambda_i = ip_i / \sum_{j=1}^{L} jp_j$ be the probability that a uniformly chosen edge is attached to a left node of degree i. Similarly, for all $i = 1, \ldots, R$, let $\rho_i = iq_i / \sum_{j=1}^{R} jq_j$ be the probability that a uniformly chosen edge is attached to a right node of degree i. Then, the distribution on the shape of G_ℓ is as described above for a randomly chosen Or-And tree with the following parameters: The number of children of an "OR" node is $i-1$ with probability λ_i, for all $i = 1, \ldots, L$; The number of children of an "AND" node is $i-1$ with probability ρ_i, for all $i = 1, \ldots, R$.

Consider a process where at the start each left node in the graph is labeled with 0 initially with probability δ, and is labeled with 1 with probability $1-\delta$. This corresponds to missing a random fraction δ of the message bits. The goal is to eliminate as many as possible 0 labels according to the graph recovery rule described in the previous subsection, i.e., to recover many of the missing message bits as possible using the simple decoding process.

Let's analyze the probability y_ℓ that the left node v of a uniformly chosen edge (v, w) is labeled with 0 considering the process running only on the subgraph G_ℓ. (It is clear that v will definitely change its label to 1 in the process running on the entire graph if it does so with process running just on G_ℓ.) Note that v obtains the label 1 with respect to G_1 if it is either received directly (with probability $1-\delta$, or if for at least one of its right neighbors w' $(w \neq w')$, all left neighbors of w' excluding v are received directly. Note that v has $i-1$ right children excluding w with probability λ_i, and that for any child w' of v, w' has $i-1$ left children excluding v with probability ρ_i. Define the polynomials

$$\lambda(x) = \sum_{i=1}^{L} \lambda_i \cdot x^{i-1}, \text{ and} \tag{4}$$

$$\rho(x) = \sum_{i=1}^{R} \rho_i \cdot x^{i-1}. \tag{5}$$

Then, from Lemma 1, $y_\ell = \delta \cdot \lambda(1 - \rho(1 - y_{\ell-1}))$. Using this equation, the probability that v has label 0 with respect to G_ℓ can be computed iteratively starting with the equation $y_0 = \delta$. We would like that y_ℓ goes to 0 as ℓ grows. This condition will be true if $\delta \cdot \lambda(1 - \rho(1 - x)) < x$ for all $x \in (0, \delta]$, or more precisely, there is a constant $\epsilon > 0$ such that

$$\delta \cdot \lambda(1 - \rho(1 - x)) < x(1 - \epsilon) \tag{6}$$

for all $x \in (0, \delta]$. This turns out to be equivalent to the condition given in [9] for this process to end successfully.

2.3 The Overall Analysis

It is not hard to argue that the process terminates with all message values successfully recovered if the probability a message bit is not directly received is δ and if Condition (6) is fulfilled. However, the details are somewhat tedious and thus we only sketch the proof here.

The first thing to see is that if ℓ is set to c/ϵ for some $c > 1$ then $y_\ell \leq \delta/\exp(c)$. Thus, for any constant $\gamma > 0$ we can set ℓ to a constant so that $y_\ell < \gamma$. If ℓ is a constant then the number of nodes in the graph G_ℓ is also a constant. Using this, and a standard generalization of Markov's inequality to argue about large moments of a distribution, it is not hard to argue that the number of message bits not recovered at the end of the decoding process is greater than $\gamma' n$ with probability exponentially small in $n^{\epsilon'}$, for $\gamma' \approx \gamma$ and for a constant $\epsilon' > 0$. Then, using the expansion properties of the random graph, using standard combinatorial arguments as outlined in [9], it is not hard to argue that if at most $\gamma' n$ message bits are left recovered then the decoding process fails to recover more than $O(n^{\gamma''})$ message bits with probability at most inverse polynomial in n, for some constant $\gamma'' < 1$. Finally, by a small supplement to the graph (adding a few nodes on the right and having three additional edges out of each node on the left mapping randomly to these few additional right nodes), it is not hard to show that if the process fails to recover at most $O(n^{\gamma''})$ of the message bits in the original graph then in the supplemented graph all message bits fail to be recovered with probability at most inverse polynomial in n. From this it follows that with high probability the decoding process terminates when all message bits have been successfully recovered.

2.4 Extensions Using the New Analysis

There are a couple of extensions of the decoding process analysis that help to overcome the following practical problems in the design of loss-resilient codes.

In the original analysis it is assumed inductively that all the check bits are received when trying to recover the message bits. The reason we made this assumption is that in the original construction the cascading sequence of bipartite graphs is completed by adding a standard loss-resilient code at the last level.

There are some practical problems with this. One problem is that it is inconvenient to combine two different types of codes. A more serious problem is that standard loss-resilient codes take quadratic time to encode and decode. Suppose the message is stretched to an encoding twice its length. In order to have the combined code run in linear time, this implies that the last graph in the cascading sequence has $\sqrt{n}$ left nodes, where n is the number of nodes associated with the original message, i.e., there are $\log(n)/2$ graphs in the sequence. In the analysis, we assume that an equal fraction of the nodes in each level of the graph is received. However, there is variance in this fraction at each level, with the worst expected fractional variance at the last level of $1/\sqrt[4]{n}$. Thus, if a message of length $65,536$ is stretched to an encoding of length $131,072$, then just because of the variance of $1/\sqrt[4]{n} = 0.063$, we expect to have to receive 1.063 times the message length of the encoding in order to recover the message.

The solution to this problem is to use many fewer levels of graphs in the cascade, and at the last level also use a graph. We have tried this idea, with the last graph chosen from an appropriate distribution, and it works quite well. For example, using only three levels of graphs we can reliably recover a message of length $65,536$ from a random portion of length $67,700$ (i.e., 1.033 times the optimal of $65,536$) of an encoding of length $131,072$.

To design the graph for this solution, we need to analyze the decoding process when a random portion of both the message bits and the check bits are missing. With the Or-And tree analysis, this is straightforward. Recall the terminology established in Subsection 2.2. Suppose that a random fraction δ of the message bits are not received directly and a random fraction δ' of the check bits are not received directly. The generalization of Condition (6) for this case when there are losses on both sides is that the process terminates in a state where a uniformly chosen edge is adjacent to left node with a missing message bit with probability at most γ if there is a constant $\epsilon > 0$ such that

$$\delta \cdot \lambda(1 - (1 - \delta') \cdot \rho(1 - x)) < x(1 - \epsilon) \tag{7}$$

for all $x \in (\gamma, \delta]$.

In [9] a procedure is described for finding (close to) optimal right probabilities $\rho_1, \ldots, \rho_R$ for a given set of left probabilities $\lambda_1, \ldots, \lambda_L$ using a linear programming approach. However, [9] did not describe how to find the optimal left probabilities for a given set of right probabilities. Using Condition (6) (or, even stronger, Condition (7), it is easy to see how to use the methodology described in [9] to do exactly this. In fact, Condition (6) is in some sense of the dual of the corresponding condition described in [9], which was

$$\rho(1 - \delta \cdot \lambda(1 - x)) > x(1 + \epsilon) \tag{8}$$

for some constant $\epsilon > 0$ and for all $x \in (0, 1]$. It is from Condition (8) that [9] shows how to find the optimal right probabilities for a given set of left probabilities. We leave it as an exercise how to use the Or-And tree analysis to easily derive Condition (8). It turns out that Condition (8) can also be derived using a couple of simple algebraic manipulations from Condition (7). We leave this as an exercise as well.

One advantage of being able to solve for both the optimal left probabilities for a given set of right probabilities and the optimal right probabilties for a given set of left probabilities is that we can invoke a "back and forth" strategy to get a good pair of distributions. The back and forth strategy consists of starting with any given set of left and right probabilities with a given average degree, and then iteratively invoke the "find the best left for the given right" followed by "find the best right for the given left". We have tried this strategy and it gives good results, although at this point we have not proved anything about its convergence to the best overall pair of probabilities.

3 Pure Literal Analysis

In this section, we consider a simple heuristic, called the *pure literal rule*, for finding a satisfying truth assignment to a boolean formula. The behavior of the pure literal rule has been studied previously with respect to randomly chosen k-SAT formula ([6], [11]). Here, we show how the tree analysis gives a direct explanation of the behavior of the pure literal rule for randomly chosen k-SAT formula with respect to the same distributions considered in [6] and [11]. With this new analysis, it is also straightforward to analyze distributions that were not previously considered and which would be much harder to analyze using previous techniques applied to this problem.

A k-SAT formula F with m clauses on n variables $\{x_1, \ldots, x_n\}$ consists of m clauses $C_1, \ldots, C_m$, each clause containing exactly k of the $2n$ possible literals $\{x_1, \bar{x}_1, \ldots, x_n, \bar{x}_n\}$. Then, the formula F is the "and" of the m clauses, and each clause is the "or" of the k literals it contains, i.e., for any $0/1$ assignment to the variables, F evaluates to 1 if and only if in each clause there is at least one literal that has value 1 with respect to the assignment. The most widely studied distribution on F is the *uniform* distribution. For fixed value of k, m, n the uniform distribution on choosing a formula F can be described as follows: For each $i \in \{1, \ldots, m\}$, for each $j \in \{1, \ldots, k\}$, each of the $2n$ possible literals is chosen with equal probability to be the j^{th} literal in clause C_i.

Let P be initially the set of all possible $2n$ literals. The *pure literal rule* heuristic for finding a satisfying assignment consists of repeated application of the following:

Pure Literal Rule: *While there is a literal $z \in P$ that appears in zero clauses, remove all clauses containing the negation $\bar{z}$ of z, assign $\bar{z}$ the value 1 (and thus z is assigned 0), and remove both z and $\bar{z}$ from P.*

The problem of interest is to study the asymptotic behavior of the pure literal rule with respect to uniformly chosen k-SAT formula for a fixed value of k, and for a fixed ratio $c = m/2n$ of the number of clauses to the number of literals, as the number of variables n grows to infinity. The more particular question is for which values of k and c will the pure literal rule almost surely (with respect to a uniformly chosen formula F) find an assignment which makes F evaluate to 1 as n goes to infinity.

Similar to the loss-resilient codes, we can describe the structure of the formula F as a bipartite graph, only in this case the edges are labelled. There are

n left nodes in the graph corresponding to the variables, and there are m right nodes corresponding to the clauses. There is an edge labelled "+" from variable x to all clauses that contain x, and there is an edge labelled "-" from variable x to all clauses that contain $\bar{x}$.

One can describe the behavior of the pure literal rule on this graph. The pure literal rule is equivalent to repeated application of the following process on this graph:

Graph Pure Literal Rule: *If there is a variable x such that all edges touching x have the same label (either all "+" or all "-") then delete x, all neighboring clauses of x, and all edges touching any of these nodes.*

The pure literal rule finds an assignment that satisfies the formula iff there are no remaining right nodes after all possible applications of this process have been made.

We describe a general way of choosing a random formula F in the terminology of the bipartite graph. Let $(p_0, p_1, \ldots, p_L)$ and $(q_0, q_1, \ldots, q_R)$ be probability vectors. Suppose each variable x chooses independently to have i edges attached to it with the same label with probability p_i, and the distribution is the same for both possible labels. Suppose each clause chooses independently to have degree j with probability p_j. For there to be the same number of edges out of the left side as there out of the right side, we require that the number of edges e satisfies

$$e = 2n \cdot \sum_{i=0}^{L} i p_i = m \cdot \sum_{j=0}^{R} j q_j.$$

A random permutation π of $\{1, \ldots, e\}$ is chosen, and then, for all $i \in \{1, \ldots, e\}$, the edge with index i out of the left side is identified with the edge with index π_i out of the right side.

For the special case of the uniform distribution on k-SAT, each clause has degree k, and the number of edges with the same label out of each variable is asymptotically distributed according to the Poisson distribution with mean $\theta = kc$ as n goes to infinity, i.e., the probability a particular literal appears in i clauses is asymptotically $\exp(-\theta) \cdot \theta^i / \theta!$.

Consider the random subgraph G_ℓ of this graph obtained as follows: Choose an edge uniformly at random from among all edges, and suppose it is an edge from variable x to clause C with label $* \in \{+, -\}$. Consider the subgraph G_ℓ obtained by the following search. Consider variable x to be at the zeroth level of the search. Follow all the edges out of x with the opposite label of $*$. This leads to a first level of clause nodes. Let C' be one of the clauses at the first level. Follow all edges out of C' except the edge that led into C'. This leads to a second level of node variables. Let $*' \in \{+, -\}$ be the label of an edge from C' to some variable x'. Follow all the edges out of x' with the opposite label of $*'$. In a similar pattern, continue this breadth first search out to level 2ℓ.

As was the case for the loss-resilient codes, G_ℓ is a tree with high probability for a fixed value of ℓ as n goes to infinity. Furthermore, asymptotically the distribution on the shape of G_ℓ can be described as follows. For all $i = 1, \ldots, L$, let $\lambda_i = ip_i / \sum_{j=1}^{L} jp_j$ be the probability that a uniformly chosen edge is attached to a variable node with i edges attached of the opposite label of the edge. Similarly, for all $i = 1, \ldots, R$, let $\rho_i = iq_i / \sum_{j=1}^{R} jq_j$ be the probability that a uniformly chosen edge is attached to a clause node of degree i. Then, the distribution on the shape of G_ℓ is as described above for a randomly chosen Or-And tree with the following parameters: The number of children of a variable node is i with probability λ_i, for all $i = 1, \ldots, L$; The number of children of a clause node is $i-1$ with probability ρ_i, for all $i = 1, \ldots, R$.

Consider the following labelling process of the nodes in the tree G_ℓ that starts at the leaves at level 2ℓ and works up towards the root at level 0. A leaf variable node is labelled with a 1 iff it would have no descendants if the tree were extended one additional level. An internal variable node is labelled with a 1 iff it either has no direct descendants or else they are all labelled with a 1. A clause node is labelled with a 1 iff at least one direct descendant is labelled with a 1. It can be checked that if the root node receives the label 1 then the pure literal rule will give that variable an assignment.

Let y_ℓ be the probability that the root node of G_ℓ will receive the label 1 by the above labelling process. Define the polynomials

$$\lambda(x) = \sum_{i=1}^{L} \lambda_i \cdot x^i, \text{ and} \tag{9}$$

$$\rho(x) = \sum_{i=1}^{R} \rho_i \cdot x^{i-1}. \tag{10}$$

From Lemma 1 it follows that this can be expressed as

$$y_\ell = \lambda(1 - \rho(1 - y_{\ell-1})) \tag{11}$$

In order for the pure literal rule to end with a complete assignment that satisfies the formula, we want all variables to disappear from the formula, which means that we want

$$\lambda(1 - \rho(1 - y)) > y \tag{12}$$

for all y in the range $[\lambda_0, 1)$ (Noting that $y_0 = \lambda_0$).

For a uniformly chosen k-SAT formula, $\lambda(x) = \exp(-\theta(x-1))$ and $\rho(x) = x^{k-1}$. By using Maple, one can verify that Condition 12 is satisfied for all y when $k = 3$ at the threshold value $\theta = 1.63$. This result has been found previously by [4] and [11] using a different approach. The advantage of the tree analysis approach employed in this paper is that it is easily possible to analyze substantially different distributions for choosing the formula with little additional difficulty.

4 Greedy Matching Analysis

In the paper [7], the analysis of a simple and fast heuristic for finding matchings was described and analyzed with respect to randomly chosen graphs. The tree analysis approach can be used to provide an alternative analysis. The details are similar to, but somewhat different than, those presented above to analyze the loss-resilient codes. As mentioned above, the advantage of the tree analysis is that it can be adopted to analyze a variety of distributions on the graph with little additional effort. For lack of space, we omit details of this analysis in this version of the paper.

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