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EE225D

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Wave Basics

Lecture 11

Main Focus - the Vibrating String and the Acoustic Tube

String:
- Violin → bowed
- Guitar → plucked
- Cello
- Piano → struck

Different Excitations

Tube:
- Trumpet
- Clarinet
- Human Voice

Summary

- String
- Tube
- Time-space digitization
- Example of ‘digitized’ pluck result
- Resonance [analog]
- Resonance [digital]

Strings (Assumption to make math. easier)

- No stiffness
- Constant tension - S
- Constant mass,density - ε
- No friction
- No gravitational effects

- Newtons 2nd Law $F_y = m a_y$

$$S\frac{\partial^2 y}{\partial x^2} = \varepsilon dx \qquad \frac{\partial^2 y}{\partial t^2}$$

Let $C = \sqrt{\frac{S}{\varepsilon}}$

$$C^2\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}$$

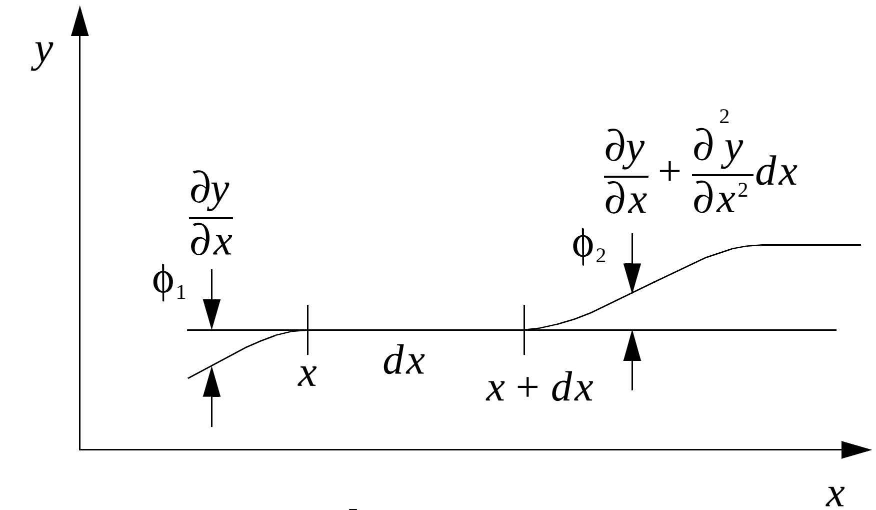


Fig.10.1 : Vibrating String Geometry.

Total Vertical Force = $S\frac{\partial^2 y}{\partial x^2}dx$

Tubes

Ideal string vibrates vertically & propogates horizontally [transverse].

But tube vibration is in the same direction as the propogation [longitudinal].

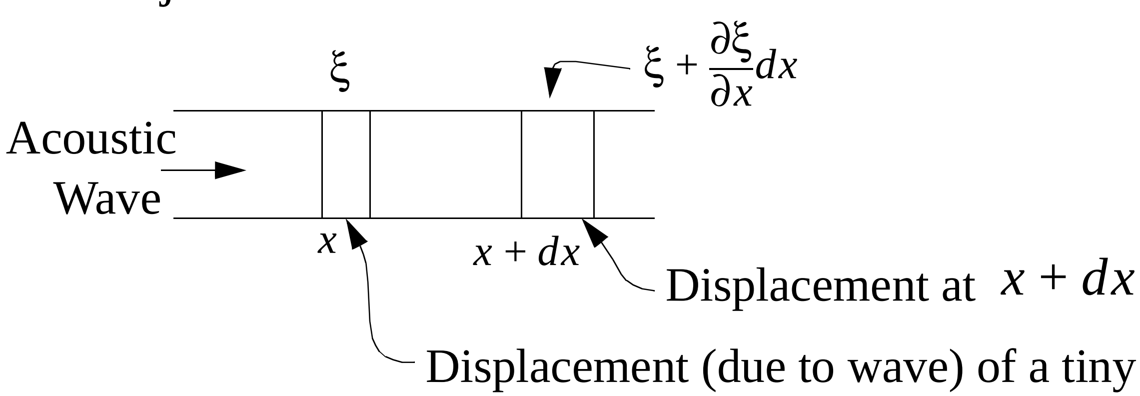
- Tube at rest [no acoustic wave].

parameters A=cross-sectional area

Variables of Interest

ρ = Air Density
p = Air Pressure
u = Air Velocity of a tiny volume

-Tube subject to wave.



$$u = \frac{\partial \xi}{\partial t}$$

Assumptions :

- Plane wave propogation for frequencies below ~4000Hz.

$c = f\lambda$ as $\lambda\uparrow$, plane wave assumption gets better.

$$\text{so } \lambda = \frac{340\text{m/s}}{3400\text{Hz}} = 0.1 \text{ meter} = 10\text{cm}.$$

- No thermal conduction losses.

- No viscosity losses.

- Rigid walls.

- Cross-section area constant.

Using

- Newton’s 2nd Law
- Mass Conservation
- Assume that pressure change is proportional to air density change

$$c^2\frac{\partial^2 p}{\partial x^2} = \frac{\partial^2 p}{\partial t^2}$$

$$c^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

$$c^2 = \sqrt{\gamma\frac{\rho_o}{\rho_o}}$$

$$\gamma = \frac{\text{spec. ht. of gas at constant pressure}}{\text{spec. ht. at constant volume}}$$

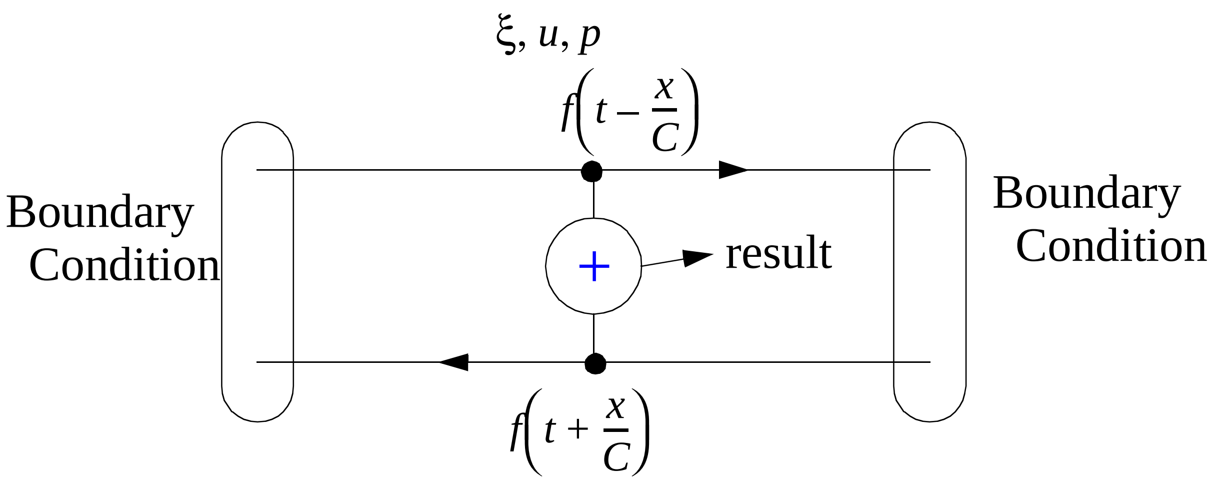
$$\frac{\partial^2 f}{\partial t^2} = \ddot{f}\left(t - \frac{x}{c}\right) \qquad \frac{\partial f}{\partial t} = f\left(t - \frac{x}{c}\right) = \frac{\partial^2 f}{\partial t^2}$$

$$\frac{\partial^2 f}{\partial x^2} = \ddot{f}\left(t - \frac{x}{c}\right)\left(\frac{1}{c^2}\right) \qquad \frac{\partial f}{\partial x} = -\frac{1}{c} f\left(t - \frac{x}{c}\right), \frac{\partial^2 f}{\partial x^2} = f\left(t - \frac{x}{c}\right)\left(\frac{1}{c^2}\right)$$

$$\therefore f\left(t - \frac{x}{c}\right) = \boxed{c^2 \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial t^2}}$$

$f\left(t + \frac{x}{c}\right)$ also satisfies the wave equation.

So the solution can be the sum or difference of waves travelling in opposite directions



At any time t,
the result is a function of x.

At any point x,
the result is a function of t.

Let's unite equations in terms of $u(x, t) = \frac{\partial\xi}{\partial t}(x, t)$

$$u(x, t) = u^{+}\left(t - \frac{x}{c}\right) - u^{-}\left(t + \frac{x}{c}\right)$$

$$p(x, t) = Z_O\left\{u^{+}\left(t - \frac{x}{c}\right) + u^{+}\left(t + \frac{x}{c}\right)\right\}$$

To find Z_O, $\frac{\partial p}{\partial x} = -\frac{\rho\partial u}{A\partial t}$ $(F = ma)$ $\longrightarrow$ $Z_O = \frac{\rho c}{A}$

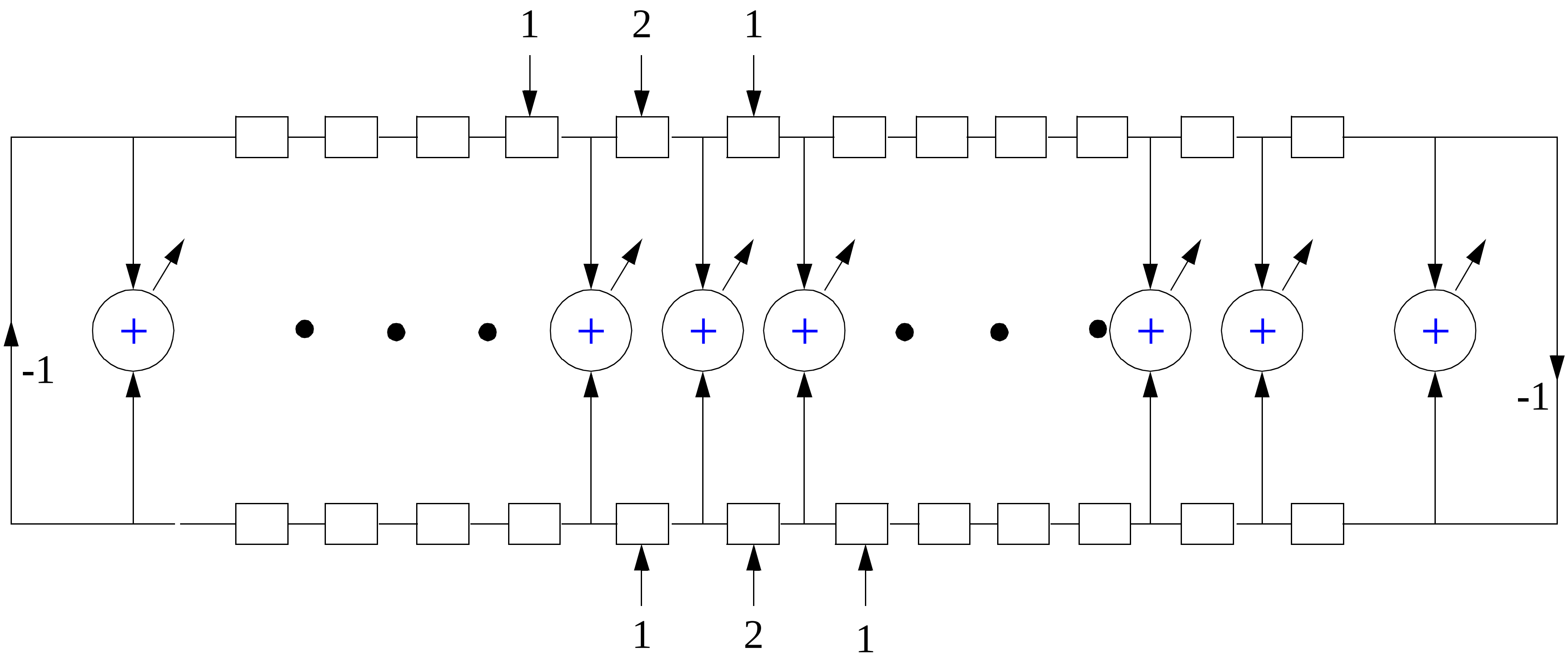


Figure 10.2 : Discrete-time simulation of a string fixed at both ends.

$x = 0$ (leftmost sum) and $x = l$ (rightmost sum), $y(x, t) = 0$.

Consider a finction of m and n.

n is (discrete) time.

m is (discrete) space.

T is sampling interval in time.

X is sampling interval in space.

Equation 10.10 : $y(mX, nT) = Af(mX - cnT) + Bf(mX + cnT)$

A and B depend on boundary conditions specified in time and space.

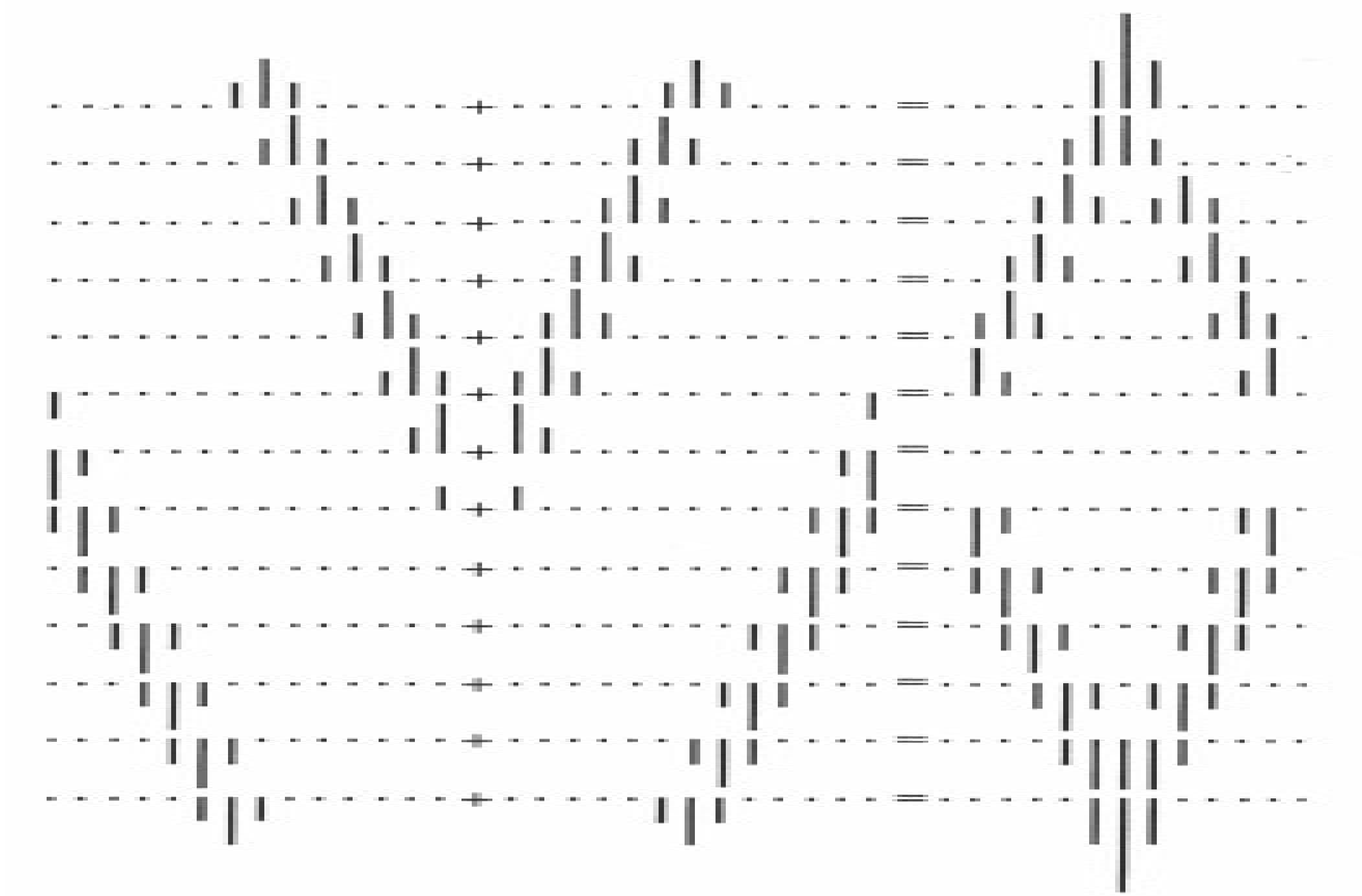


Figure 10.3 : Step by step history of travelling waves stimulated by a pluck.

Example - Fig. 10.3:

- String is plucked.

line 0 - At t=0, string is released and allowed to vibrate.

- Because of symmetry, propogation is set up in both $\pm$ directions.

line 1 - Right hand condition has moved one step left.

- Left hand condition has moved one step right.

- Sum is shown at right.

line 5 - Sum has gone to zero at both ends.

This can only happen if left hand and right hand fictitions patterns are as shown in the figure.

By line 8, propagation directions in both right and left half have been reversed.

Problem 10.2 asks you to continue the sequence until it has returned to the initial condition.

Since the sequence has been assumed lossless, it will go thru the same cycle as before.

Thus, the space-time function is periodic in time.

This means that a fourier representation can work.

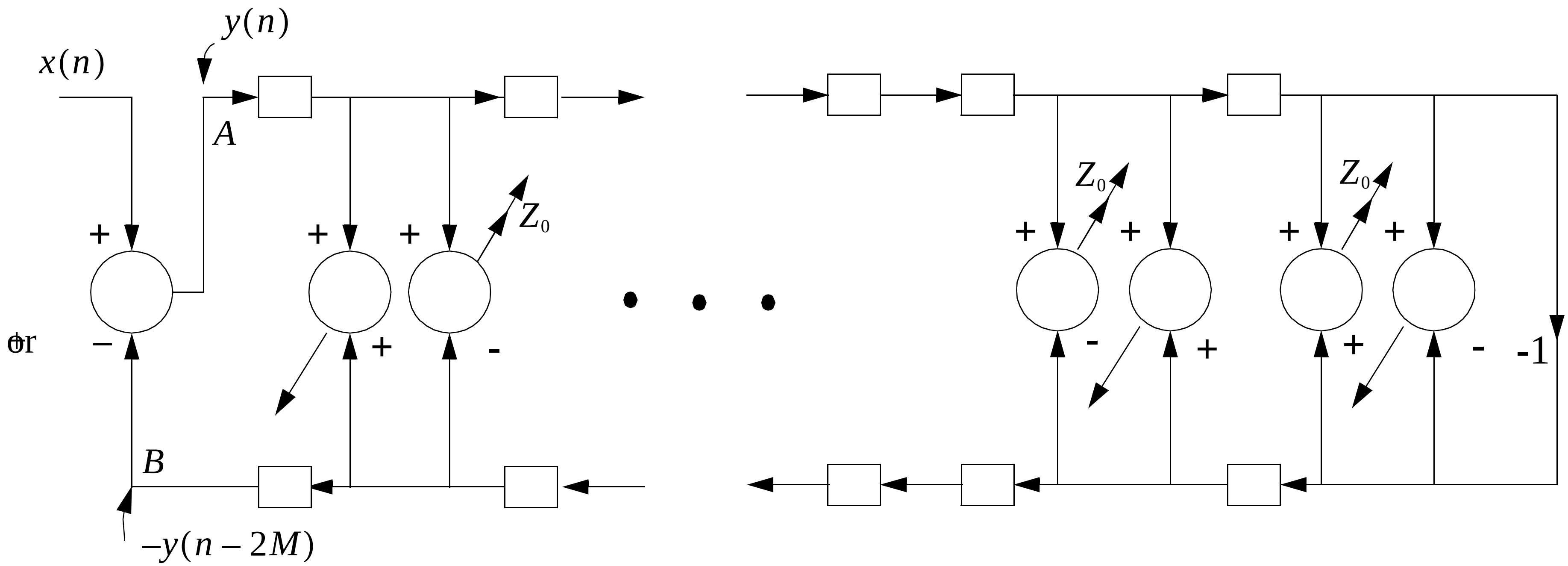


Figure 10.5 : Discrete-time simulation of pressure and volume velocity in an acoustic tube. Up arrows are pressure waves, down arrows are velocity waves, and small squares are unit delays. The plus sign on the left corresponds to a tube open at the left, and the minus sign signifies a closed tube at the left. The tube is always closed at the right.

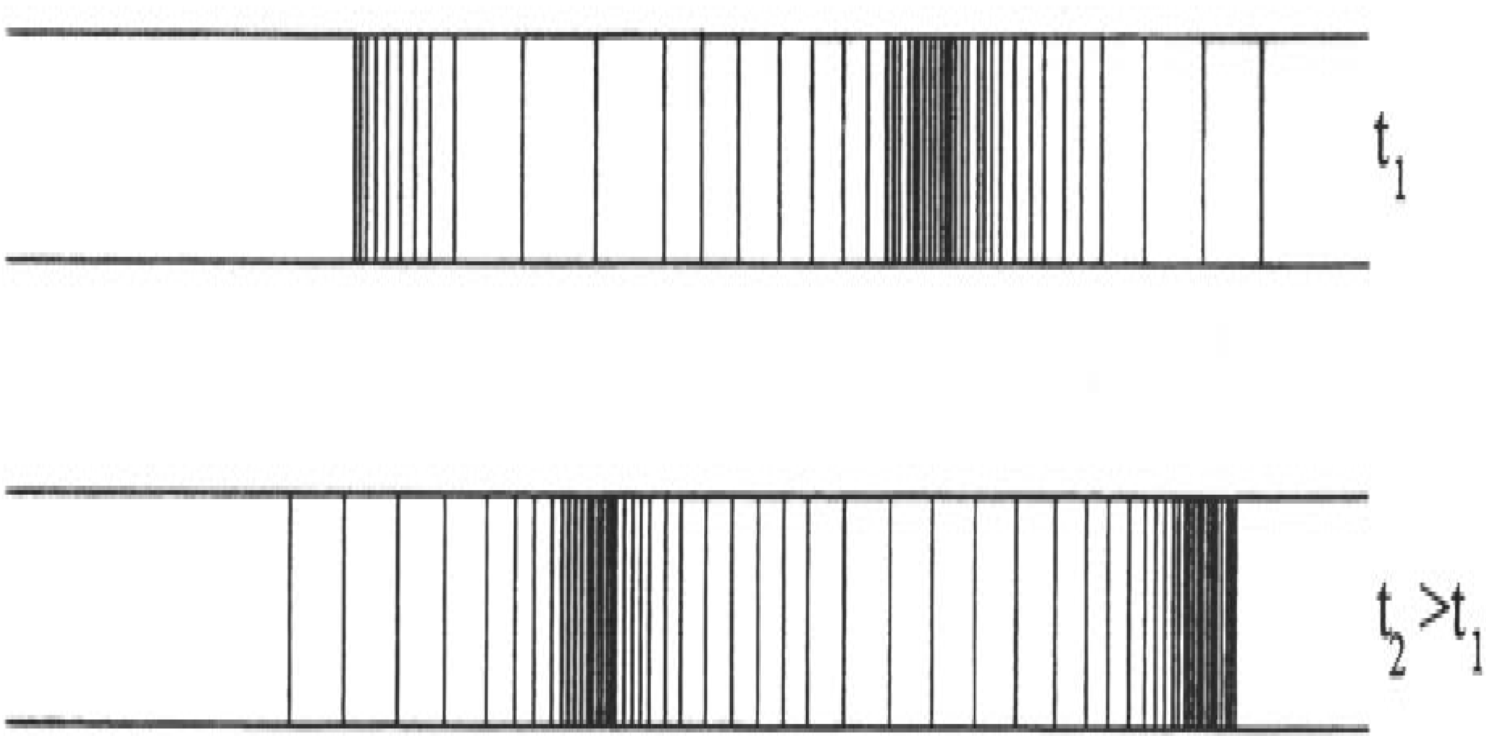
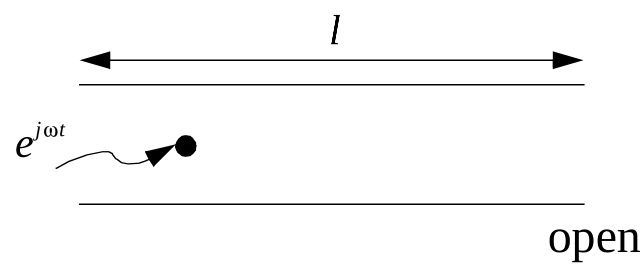


Figure 10.4 : Longitudinal waves in an air column.

Resonance in Acoustic Tubes



$p(l, t) \equiv 0$

Velocity $$u(x, t) = u^{+}\left(t - \frac{x}{C}\right) - u^{-}\left(t + \frac{x}{C}\right)$$

Pressure $$p(x, t) = Z_o\left(u^{+}\left(t - \frac{x}{C}\right) + u^{-}\left(t - \frac{x}{C}\right)\right)$$

For exponential input, both u^{+} and u^{-} will be exponential.

$$u^{+}\left(t - \frac{x}{C}\right) = Ae^{j\omega\left(t - \frac{x}{C}\right)} \qquad u^{-}\left(t + \frac{x}{C}\right) = Be^{j\omega\left(t + \frac{x}{C}\right)}$$

Problem - Find A & B to match boundary conditions.

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$$u(0, t) = e^{j\omega t} = Ae^{j\omega\left(t - \frac{x}{C}\right)} - Be^{j\omega\left(t + \frac{x}{C}\right)}$$

$$p(l, t) = 0 = Ae^{j\omega\left(t - \frac{x}{C}\right)} - Be^{j\omega\left(t + \frac{x}{C}\right)}$$

A & B can be found [Eliminate t].

Now, you can get result [Equation 10.24].

$$u(x, t) = \frac{\cos[(\omega(l - x))/C]}{\cos\frac{\omega l}{C}} e^{j\omega t}$$

Poles occur at $\frac{\omega l}{C} = \frac{\Pi}{2}, \frac{3\Pi}{2} \ldots \frac{(2n+1)\Pi}{2}$

$n = 0, 1, \ldots$

Final Result

$$f_n = \frac{(2n+1)C}{4l}, n = 0, 1, 2, \ldots$$

Example : Human vocal tract, during phonation of neutral vowel.

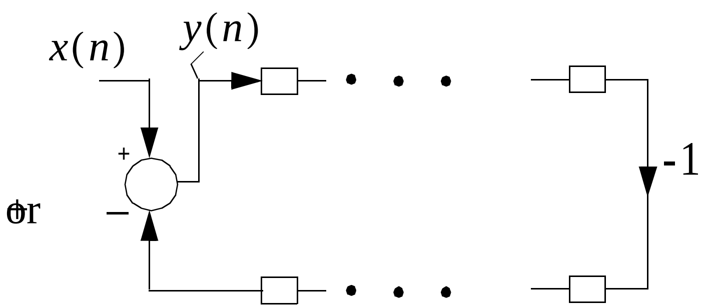
[Vocal tract looks like an open Acoustic Tube.]

$c = 340\text{m/s} \qquad l = 15\text{cm}$

$$Z_o = \frac{340\text{m/s}}{0.68\text{m}} \cong 500\text{Hz}, \; f_1 = 1500\text{Hz}, \; f_2 = 2500\text{Hz}, \text{ etc.}$$

These are close to <u>measured</u> resonances.

We can get an analogous result from the discrete-time & space simulation.



Pressure or Volume Velocity.

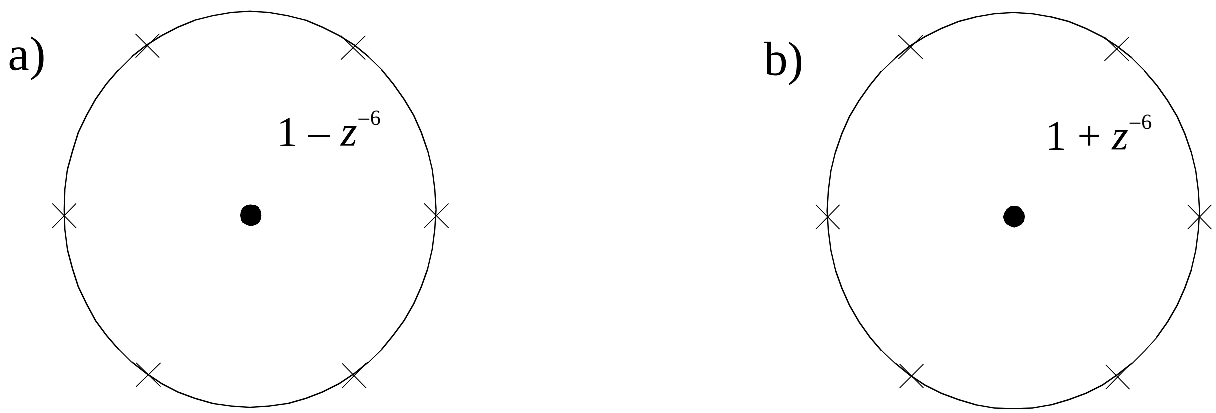
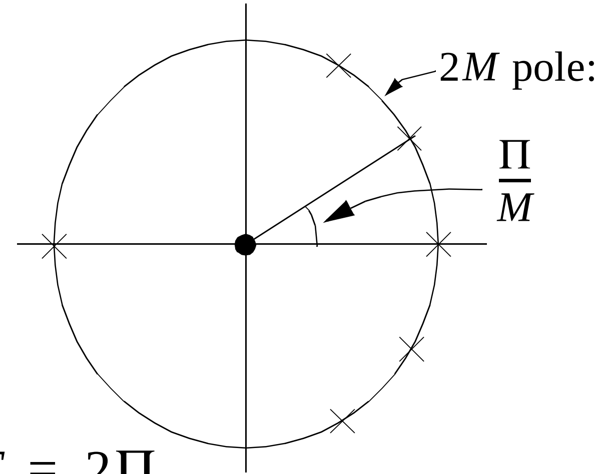


Figure 10.6 : Poles of a discrete-time simulation of a lossless, uniform acoustic tube.



$$H(z) = \frac{X(z)}{1 \mp Z^{-2N}}$$

at $Z = e^{j\omega T}$

so $Z^{-2M} = e^{-2j\omega MT}$, poles occur when $2\varpi MT = 2\Pi$

$$2\Pi f MT = 1 \qquad \boxed{f = \frac{1}{2MT}}$$

$$2\Pi fn = \frac{(2n+1)\Pi}{2MT} \qquad f_n = \frac{(2n+1)}{4MT} = \frac{(2n+1)}{4l}$$

$$\boxed{MT = \frac{l}{C}}$$

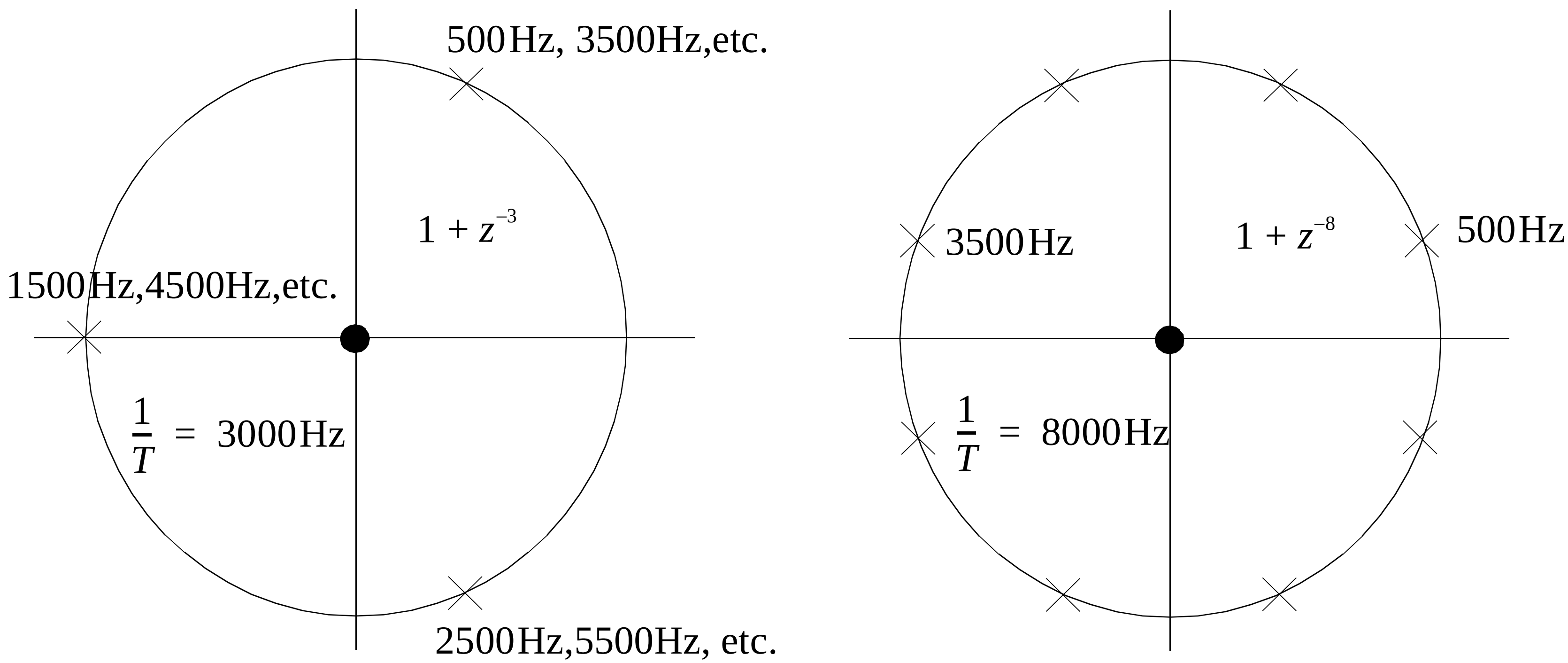


Figure 10.8 : Two examples showing that the same resonant frequencies can be obtained with different sampling rates and different spatial quantization.

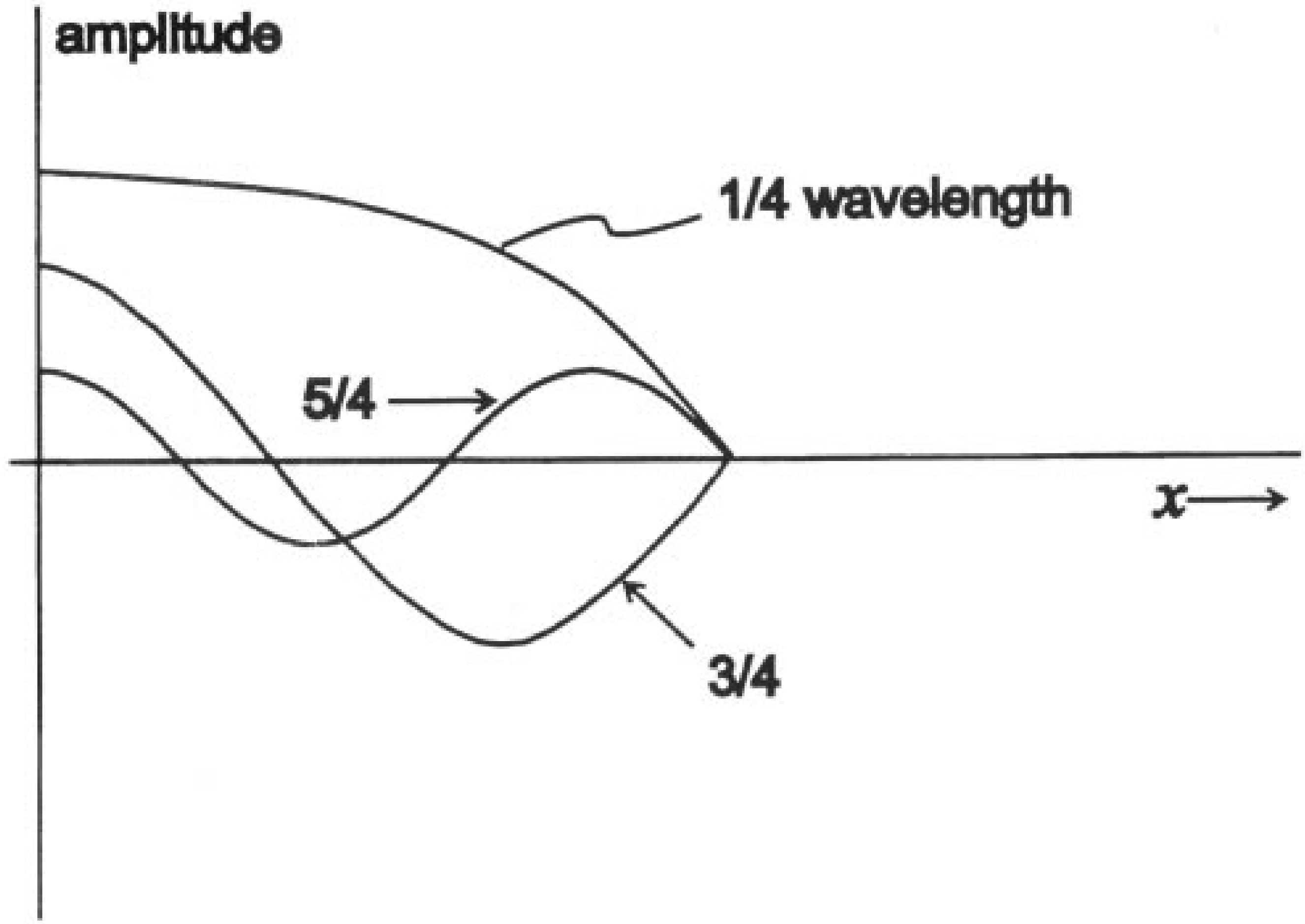


Figure 10.7 : First three models of an acoustic tube open at one end.