Matrix-free Interior Point Method for Compressed Sensing Problems

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Existing solvers

1. FPC_AS by Wen, Yin, Goldfarb, Y. Zhang, H. Zhang.
2. SPGL1 by van den Berg, Friedlander.
3. GPSR by Figueiredo, Nowak, Wright
4. $\ell_1\_\ell_s$ by Kim, Koh, Lustig, Boyd, Gorinevsky.
5. PDCO by Saunders.
6. many others ...

Outline

Objects & terminology

$x \in \mathbb{R}^n$: sparse signal

$b \in \mathbb{R}^m$: noiseless measured signal

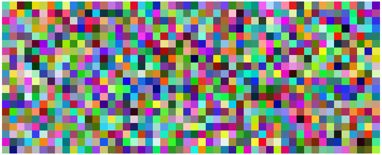
$e \in \mathbb{R}^m$: vector of noise

$\tilde{b} \in \mathbb{R}^m$: noisy measured signal

$A \in \mathbb{R}^{m \times n}$: measurement matrix

Objects and their properties

- x: $\|x\|_0 = k \ll n$ nonzero components.
- A: Satisfies the Restricted Isometry Property (RIP).

$A =$ 

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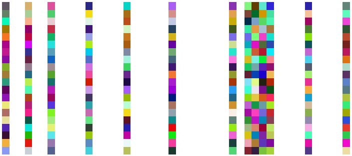
$A =$ 

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$\bar{A} =$ 

RIP

$$\left\| \frac{n}{m} \bar{A}^T \bar{A} - I_m \right\|_2 \leq \delta_k$$

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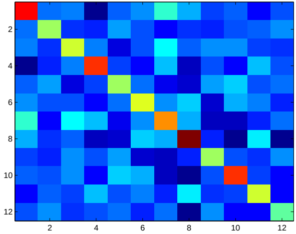


Objective of CS

Find the k-sparse $x \in \mathbb{R}^n$, having $b \in \mathbb{R}^m$ with $m \ll n$.

Noiseless measurements b

BP:
$$\min_{x \in \mathbb{R}^n} \quad \|x\|_1 \qquad \textit{subject to:} \quad Ax = b$$

Noisy measurements $\tilde{b}$

BPDN:
$$\min_{x \in \mathbb{R}^n} \quad \tau \|x\|_1 + \frac{1}{2}\|Ax - \tilde{b}\|_2^2$$

BPDN for IPMs

$$\textbf{BPDN:} \quad \min_{x \in \mathbb{R}^n} \quad \tau \|x\|_1 + \frac{1}{2}\|Ax - \tilde{b}\|_2^2$$

replace	with	using
$\|x\|_1$	$\|x\|_1 = 1_{2n}^T z$ $z \in \mathbb{R}^{2n} \geq 0$	$\|x_i\| = z_i + z_{i+n}$ $z_i \geq 0, \ \forall i$
$\|Ax - \tilde{b}\|_2^2$	$F\tilde{b} + z^T F F^T z$	$x_i = z_i - z_{i+n}$

$$F = \begin{bmatrix} A^T \\ -A^T \end{bmatrix} \in \mathbb{R}^{2n \times m}$$

$$FF^T = B = \begin{bmatrix} A^T \\ -A^T \end{bmatrix} \begin{bmatrix} A & -A \end{bmatrix} = \begin{bmatrix} A^T A & -A^T A \\ -A^T A & A^T A \end{bmatrix} \in \mathbb{R}^{2n \times 2n}$$

Primal-Dual BPDN Programs

Primal BPDN

$$\min_{z \in \mathbb{R}^{2n}} \quad c^T z + \frac{1}{2} z^T B^T z$$
$$\textit{subject to:} \quad z \geq 0$$

Dual BPDN

$$\max_{z,s \in \mathbb{R}^{2n}} \quad -\frac{1}{2} z^T B^T z$$
$$\textit{subject to:} \quad B^T z - s = -c$$
$$z, s \geq 0$$

$$B = \begin{bmatrix} A^T A & -A^T A \\ -A^T A & A^T A \end{bmatrix} \in \mathbb{R}^{2n \times 2n}, \quad c = \begin{bmatrix} \tau 1_n - A^T \tilde{b} \\ \tau 1_n + A^T \tilde{b} \end{bmatrix} \in \mathbb{R}^{2n}$$

Matrix-free IPM

Calculate the direction by solving:

Reduced Newton system for Primal-Dual BPDN

$$(B + \Theta^{-1}) \times \Delta z = *$$

$\Theta \in \mathbb{R}^{2n \times 2n} = S^{-1}Z$ and $Z, S \in \mathbb{R}^{2n \times 2n}$ are diagonal matrices

Matrix-free Regime

- Inexact solution using an iterative process, i.e CG.
- Only matrix-vector products are allowed.
- If the matrix $B + \Theta^{-1}$ is an operator then the process is memoryless.

Solution of linear systems for IPMs

Use the CG method to solve linear systems with the matrix:

$$M = B + \Theta^{-1} = \begin{bmatrix} A^T A & -A^T A \\ -A^T A & A^T A \end{bmatrix} + \begin{bmatrix} \Theta_1^{-1} & \\ & \Theta_2^{-1} \end{bmatrix}$$

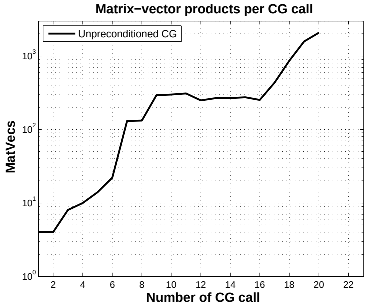
$\Theta_i \in \mathbb{R}^{n \times n}$, $i = 1, 2$, are the (i,i) blocks of $\Theta = S^{-1} Z$.

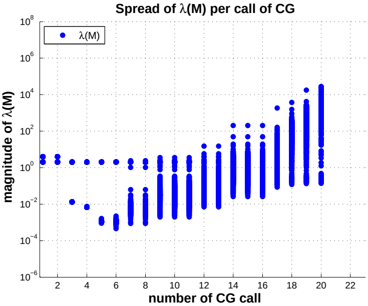
The fast convergence of the CG method depends on:

- The clustering of $\lambda(M)$. Better clustering $\Rightarrow$ faster CG.

How the $\lambda(M)$ behave as the IPM progresses?

Slow convergence of the CG for IPMs





Required accuracy: 1.0e-6

Objectives of preconditioning

Reduce the computational efforts of the CG method with preconditioning.

- Introduce a matrix $P \in \mathbb{R}^{2n \times 2n}$ and solve instead:

$$P^{-1}M = P^{-1}*$$

An efficient preconditioner should:

1. Be easily invertible.
2. Cluster the eigenvalues $\lambda(P^{-1}M)$

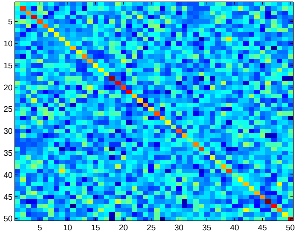
Proposed preconditioner

Approximate M:

$$\mathrm{M} = \begin{bmatrix} A^TA & -A^TA \\ -A^TA & A^TA \end{bmatrix} + \begin{bmatrix} \Theta_1^{-1} & \\ & \Theta_2^{-1} \end{bmatrix}$$

with P:

$$\mathrm{P} = \rho \begin{bmatrix} \mathrm{I} & -\mathrm{I} \\ -\mathrm{I} & \mathrm{I} \end{bmatrix} + \begin{bmatrix} \Theta_1^{-1} & \\ & \Theta_2^{-1} \end{bmatrix}$$

$A^TA =$ 

k entries of $\Theta^{-1} \to 0$

$2n - k$ entries of $\Theta^{-1} \to \infty$

Spectral properties of $P^{-1}M$

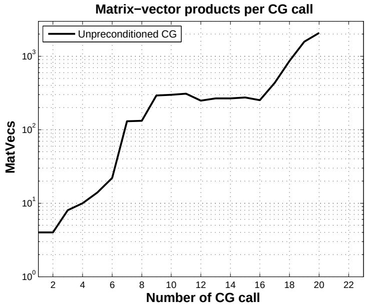
Theorem

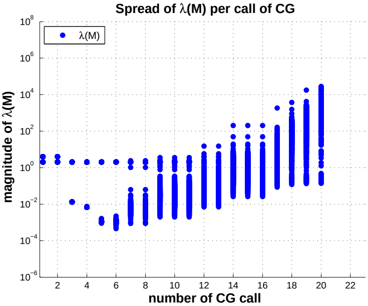
- Exactly n eigenvalues of $P^{-1}M$ are 1.
- The remaining n satisfy $|\lambda(P^{-1}M) - 1| \leq \delta_k + \frac{n}{m\delta_k L}$

where $L = \mathcal{O}(\max_i(\Theta_1 + \Theta_2)^{-1}) \to \infty$ and δ_k is the RIP-constant.

1. Universality of the proof! It holds $\forall A \in \mathbb{R}^{m\times n} \vdash$ RIP.
2. P^{-1} works if $\|x\|_0 \leq$ maximum k such that $A \vdash$ RIP.
3. P^{-1} is not efficient if the problem is unsolvable.

Practical performance





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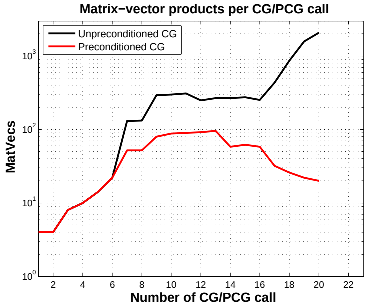

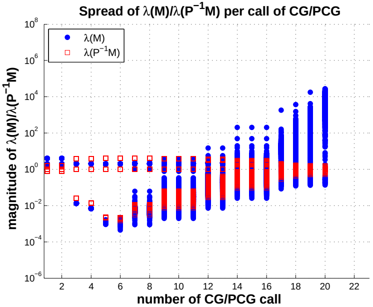

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Results I: Sparco test suite

Table : Number of matrix-vector products

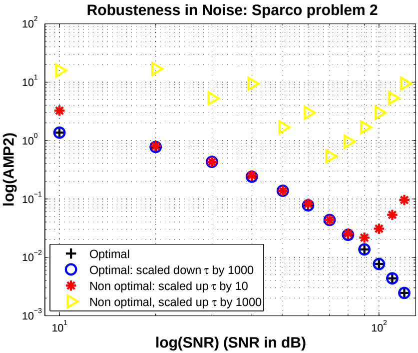
ID	rhs	Accuracy	mfipm	ℓ_1_ℓ_s	pdco	fpc_as cg	spgl1
2	$\tilde{b}$	3.0e-04	61	48	687	9	40000
	b	1.0e-11	65	98	40007	40002	22
3	$\tilde{b}$	7.0e-04	241	462	4941	106	40000
	b	1.0e-08	415	1612	40157	212	148
5	$\tilde{b}$	2.0e-03	5991	9842	28203	521	40000
	b	2.0e-05	7953	19684	41283	874	2567
7	$\tilde{b}$	4.0e-03	179	272	425	62	39
	b	1.0e-06	255	850	601	76	81
9	$\tilde{b}$	1.0e-03	689	1546	7065	1680	40000
	b	5.0e-12	649	1886	6845	40016	40000
10	$\tilde{b}$	1.0e-03	4775	8529	6203	40002	40000
	b	9.0e-10	4567	8192	41227	40161	40000
701	$\tilde{b}$	2.0e-02	947	1794	5967	1049	40000
	b	7.0e-09	1341	2656	42041	40017	15239
702	$\tilde{b}$	4.0e-03	809	1574	3341	40001	40000
	b	1.0e-07	1123	3030	49563	40157	11089
902	$\tilde{b}$	3.0e-04	181	588	261	40	40000
	b	2.0e-06	225	675	275	42	59
903	$\tilde{b}$	7.0e-03	2201	4944	5939	11395	8941
	b	3.0e-06	4173	25114	29901	40161	40000

Noise robustness

- SNR= 10, 20, ..., 120 dB.
- The quality of reconstruction is measured by the amplification factor:

$$\text{AMP2} = \frac{\sqrt{\frac{1}{n}\|x - \hat{x}\|_2}}{\sqrt{\frac{1}{m}\|e\|_2}}$$

- The FPC_AS CG solver is used as a reference algorithm which produces the minimum AMP2.



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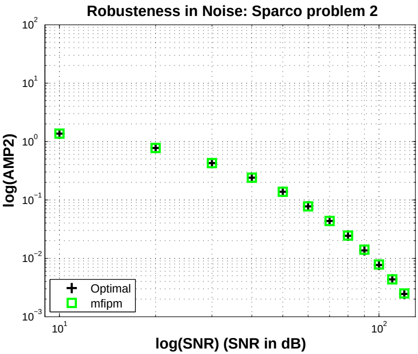
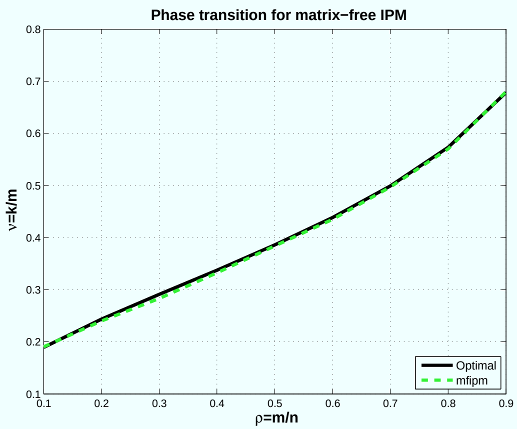


Optimal phase transition



$A \in \mathbb{R}^{m \times n}$: partial Discrete Cosine Transform.

$x \in \mathbb{R}^n$: randomly placed ± 1's at $k \ll n$ positions.

Conclusions

- Matrix-free IPM.
- Implements universal and robust preconditioner.
- Robust reconstruction for noisy measurements.
- Faster on some signal processing applications.
- Optimal phase transition.

Further research

- Substantial decrement of the computational complexity via reduction of Newton's systems size (i.e fewer rows/columns).
  - Pros: The computational complexity will be greatly reduced.
  - Cons: The complexity of the matrix-free IPM will increase.
- Revision of the MATLAB software
  - Implementation of sparse directions.
  - Exploit k-sparse operators, i.e sparse FFT `http://groups.csail.mit.edu/netmit/sFFT/`.
  - Implementation of regularization techniques.
- Extend to $\ell_1 - \textit{analysis}$ ($\|Wx\|_1$) and total variation ($\|x\|_{TV}$).

Thank you!

Kimon Fountoulakis, Jacek Gondzio, and Pavel Zhlobich.
Matrix-free interior point method for compressed sensing problems.
Technical report, School of Mathematics, The University of Edinburgh, 2012.
Submitted to Mathematical Programming Computation.