Promise problem: positive and negative instances

Promise problem: $(P, N)$

for some $P, N \subseteq \Sigma^*$
Given as input:

\[ x \in (P \cup N) \]

Is

\[ x \in P \]

? 

Goal is to separate positive and negative instances

\[ (P, N) \in pP \]

IF

\[ M(x) \text{ halts in } \leq p(|x|) \text{ steps} \]

\[ x \in P \Rightarrow M(x) \text{ accepts} \]

\[ x \in N \Rightarrow M(x) \text{ rejects} \]

What is an ordinary language as a promise problem?
Restating SAT

\[ \text{Assign}_\phi = \{ x \in \{0, 1\}^n \mid \phi(x) = 1 \} \]

\[ \phi \in \text{SAT} \iff |\text{Assign}_\phi| \geq 1 \]

Variant of SAT: many satisfiable assignments
**Dense-Sat = (P, N)**

\[ P = \{ \phi \mid |\text{Assign}_\phi| \geq 2^{n-2}, \phi \text{ has } n \text{ variables} \} \]

\[ N = \{ \phi \mid |\text{Assign}_\phi| = 0 \} \]

What’s different about Dense-SAT and SAT?

\[ \text{Dense-Sat} \in \text{pP?} \]
How do we leverage the difference between Dense-SAT and SAT?

May have to try

$$2^n - \frac{2^n}{4} + 1 = \frac{3}{4} \cdot 2^n + 1$$

assignments

How do we solve Dense-SAT fast?

Pick a random assignment
$M(\phi)$
Pick bits $b_1, \ldots, b_n$ randomly
If $\phi(b_1, \ldots, b_n) = 1$, accept
Else reject

How fast is our algorithm?

$\text{AccPr}_M(\phi) = \Pr[M(\phi) \text{ accepts}]$

$\text{AccPr}_M(\phi) = \frac{|\text{Assign}_\phi|}{2^n}$
\[ \phi \in P \Rightarrow \text{AccPr}_M(\phi) \geq \frac{2^n - 2}{2^n} \]

\[ \phi \in N \Rightarrow \text{AccPr}_M(\phi) = 0 \]

How good was our algorithm, then?

\[ M(\phi) \]

For \( i = 1, \ldots, 300 \) do

Pick bits \( b_{i,1}, \ldots, b_{i,n} \) randomly

If \( \phi(b_{i,1}, \ldots, b_{i,n}) = 1 \), accept

Else reject

\[ \phi \in P \Rightarrow \text{AccPr}_M(\phi) \geq 1 - \left( \frac{3}{4} \right)^{300} \geq 1 - \left( \frac{1}{2} \right)^{100} \]

\[ \phi \in N \Rightarrow \text{AccPr}_M(\phi) = 0 \]
How good was our algorithm, then?

Random TM model: 4 tapes
Input tape (read-only)
Random tape (read-only)
Work tape (read-write)
Output tape (write-only)

TM $M(x; R)$
\( R \leftarrow \{0, 1\}^r(|x|) \)

What’s the limit on the random tape?

Random tape causes different outcomes for the machine’s execution

AccSet\(_M(x) = \{ R \in \{0, 1\}^{r(|x|)} \mid M(x; R) \text{ accepts}\} \)
\[
\text{AccPr}_M(x) = \frac{|\text{AccSet}_M(x)|}{2^r(|x|)}
\]

Random TMs: take an extra (unspecial) random string as input.

Let \( M \) be a poly-time TM.

\[ L \in \text{RP} \]

if

\[ x \in L \Rightarrow \text{AccPr}_M(x) \geq \frac{1}{2} \]

\[ x \notin L \Rightarrow \text{AccPr}_M(x) = 0 \]

How do we make an RP TM useful?
$P \subseteq \mathsf{RP}$

Let $\text{TM } M', L(M') \in \mathsf{P}$

$\text{TM } M, L(M) \in \mathsf{P}$

$\text{Let } \text{TM } M'(x; R) = M(x), L(M') \in \mathsf{RP}$

$\mathsf{RP} \subseteq \mathsf{NP}$
Let $M$ be a poly-time TM.

$L \in \text{BPP}$

if

$x \in L \Rightarrow \text{AccPr}_M(x) \geq \frac{2}{3}$

$x \notin L \Rightarrow \text{AccPr}_M(x) \leq \frac{1}{3}$

Reading: Sipser 10.2