Quiz 4 out
Misc updates
Promise problem: positive and negative instances
Promise problem:

\[(P, N)\]

for some

\[P, N \subseteq \Sigma^*\]
Given as input:

\[ x \in (P \cup N) \]

Is

\[ x \in P \]
Goal is to separate positive and negative instances
What is an ordinary language as a promise problem?
\((P, N) \in pP\)

IF

\(M(x)\) halts in \(\leq p(|x|)\) steps

\(x \in P \implies M(x)\) accepts

\(x \in N \implies M(x)\) rejects
Restating SAT
Assign $\phi = \{ x \in \{0, 1\}^n \mid \phi(x) = 1 \}$
\( \phi \in \text{Sat} \iff |\text{Assign}_\phi| \geq 1 \)
Variant of SAT: many satisfiable assignments
Dense-Sat = (P, N)

\[ P = \{ \phi \mid |\text{Assign}_\phi| \geq 2^{n-2}, \phi \text{ has } n \text{ variables} \} \]

\[ N = \{ \phi \mid |\text{Assign}_\phi| = 0 \} \]
What’s different about Dense-SAT and SAT?
Dense-Sat ∈ pP?
How do we leverage the difference between Dense-SAT and SAT?
May have to try assignments

\[ 2^n - \frac{2^n}{4} + 1 = \frac{3}{4} \cdot 2^n + 1 \]
How do we solve Dense-SAT fast?
Pick a random assignment
$M(\phi)$

Pick bits $b_1, \ldots, b_n$ randomly

If $\phi(b_1, \ldots, b_n) = 1$, accept

Else reject
How fast is our algorithm?
$$\text{AccPr}_{M}(\phi) = \text{Pr}[M(\phi) \text{ accepts}]$$
AccPr\textsubscript{M}(\phi) = \frac{|\text{Assign}_\phi|}{2^n}
\( \phi \in P \Rightarrow \text{AccPr}_M(\phi) \geq \frac{2^n - 2}{2^n} \)

\( \phi \in N \Rightarrow \text{AccPr}_M(\phi) = 0 \)
How good was our algorithm, then?
For $i = 1, \ldots, 300$ do

Pick bits $b_{i,1}, \ldots, b_{i,n}$ randomly

If $\phi(b_{i,1}, \ldots, b_{i,n}) = 1$, accept

Else reject
\[ \phi \in P \Rightarrow \text{AccPr}_M(\phi) \geq 1 - \left( \frac{3}{4} \right)^{300} \geq 1 - \left( \frac{1}{2} \right)^{100} \]

\[ \phi \in N \Rightarrow \text{AccPr}_M(\phi) = 0 \]
How good was our algorithm, then?
TM model: 3 tapes

**Input** tape (read-only)

**Work** tape (read-write)

**Output** tape (write-only)
Random TM model: 4 tapes

**Input** tape (read-only)
**Random** tape (read-only)
**Work** tape (read-write)
**Output** tape (write-only)
TM \quad M(x; R)
$R \leftarrow \{0, 1\}^r(|x|)$
What’s the limit on the random tape?
Random tape causes different outcomes for the machine’s execution
\text{AccSet}_M(x) = \{ R \in \{0, 1\}^{r(|x|)} \mid M(x; R) \text{ accepts} \}
\[ \text{AccPr}_M(x) = \frac{|\text{AccSet}_M(x)|}{2^r(|x|)} \]
Random TMs: take an extra (unspecial) random string as input.
Let $M$ be a poly-time TM.

$L \in \text{RP}$

if

$x \in L \Rightarrow \text{AccPr}_M(x) \geq \frac{1}{2}$

$x \notin L \Rightarrow \text{AccPr}_M(x) = 0$
How do we make an RP TM useful?
P \subseteq R_P
TM $M$, $L(M) \in P$

Let TM $M'(x; R) = M(x)$, $L(M') \in RP$
RP ⊆ NP
Let $M$ be a poly-time TM.

$L \in \text{BPP}$

if

$x \in L \Rightarrow \text{AccPr}_M(x) \geq \frac{2}{3}$

$x \notin L \Rightarrow \text{AccPr}_M(x) \leq \frac{1}{3}$