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Three Methods to Share Joint Costs or Surplus

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We study cost sharing methods with variable demands of heterogeneous goods, additive in the cost function and meeting the dummy axiom. We consider four axioms: scale invariance (SI); demand monotonicity (DM); upper bound for homogeneous goods (UBH) placing a natural cap on cost shares when goods are homogeneous; average cost pricing for homogeneous goods (ACPH). The random order values based on stand alone costs are characterized by SI and DM. Serial costsharing, by DM and UB; the Aumann–Shapley pricing method, by SI and ACPH. No other combination of the four axioms is compatible with additivity and dummy. *Journal of Economic Literature* Classification Numbers: C71, D62, D63.

1. ADDITIVE COST SHARING METHODS

The division of joint costs is a central problem in accounting [56], public utility pricing [23] and management [47]. The division of joint surplus has an even broader scope, encompassing all situations modeled by a cooperative game (in characteristic form): applications include the joint production of private [41, 17, 42, 38] or public goods [6, 26], electricity pricing [22], the allocation of waiting time among the users of a computer network [44, 45, 8], the allocation of costs among users of a network [15, 14] and more.[1]

In this paper, we use the axiomatic approach to the cost sharing problem. We work in the model known as "axiomatic cost sharing with

[1] Surveys of the applications of cooperative games to the allocation of joint surplus and joint costs are in [48, 29].

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variable demands."[2] The only available information consists of a cost function of several heterogeneous and divisible outputs. Given the level of demand in each output we ask how total cost should be shared among the various outputs? The corresponding formula may use information about cost (resp. output) at any potential level of demand (resp. supply) and nothing else. The choice of a specific formula is based on normative arguments of equity and decomposability; it is a robust and easily applicable division of costs (or surplus) relying exclusively on "objective" information about the technology.

Equivalently, the model can be interpreted as a surplus sharing problem, where agents contribute inputs to a production function and share the output. There is no difference whatsoever between the cost sharing and surplus sharing interpretations of the model; all that matters is a real valued function C defined over $\mathbb{R}^n_+$ and a particular point $(q_1, ..., q_n)$ in the domain of C; the formula then divides $C(q_1, ..., q_n)$ between the n "factors." To fix ideas, we stick to the cost sharing interpretation throughout the paper (so q_i is the demand for output i and $C(q)$ is total cost for producing the vector q), keeping in mind that the whole theory can be read equivalently in the surplus-sharing context.[3]

The seminal work in axiomatic cost sharing is by Shapley [43], in a model simpler than ours in the sense that the demand of each good is binary: each variable q_i takes only the values zero or one. The two key axioms introduced by Shapley are:

(i) *Additivity*: the cost shares depend additively upon the entire cost function, and

(ii) *Dummy*: if the demand of a certain agent does not raise the overall cost whatever the demands of other agents, then this agent's cost share is zero.

The combination of additivity and dummy characterizes the family of *random order values* ([56], see Section 7) within which an additional symmetry requirement picks the Shapley value formula.

In this paper, we follow Shapley's methodology by adopting without further justifications its two fundamental postulates, additivity and dummy. If the dummy axiom is normatively compelling, the additivity axiom is not. The latter allows us to decompose a given cost function into an arbitrary sum of cost functions and compute cost shares separately; the resulting transparency and decentralizibility of the cost sharing method have been noted by virtually every author in the literature on axiomatic cost sharing

[2] Section 4 reviews this literature and contrasts this approach with the Bayesian tradition in mechanism design.

[3] Some of the examples discussed in Sections 2 and 3 use the surplus-sharing interpretation.

with variable demands; indeed almost no paper offers a nonadditive cost sharing method to our consideration.[4]

Our first observation is that, in the model where user i may demand an arbitrary amount of the divisible good i, the two axioms additivity and dummy allow for a very rich family of methods, described in our representation theorem. See Lemma 3 and Appendix 1. The goal is to explore the consequences, within this family, of four additional normative requirements. Two of them are well known—*scale invariance* (SI) and *average cost for homogeneous goods* (ACPH). The other two are new—*demand monotonicity* (*DM*) *and upper bound for homogeneous goods* (UBH). No cost sharing method meets any three of these four requirements; on the other hand three specific pairs of these axioms characterize three remarkable cost sharing methods.

Announcing the Contents

The formal results of the paper are reviewed in Section 2. Section 3 offers a detailed discussion and interpretation of the four axioms just mentioned, emphasizing the contexts in which they are or are not compelling. Section 4 reviews the literature on axiomatic cost sharing. Section 5 defines the cost sharing model and introduces the two axioms additivity and dummy. Section 6 provides a crucial representation formula for all cost sharing methods meeting additivity and dummy. Section 7 characterizes the random order values by means of demand monotonicity and scale invariance, Theorem 1. Section 8 introduces the upper bound for homogeneous goods axiom and characterizes serial cost sharing by the combination of demand monotonicity and UBH, Theorem 2. Some concluding comments are gathered in Section 9 and all proofs in the Appendix.

2. OVERVIEW OF THE PAPER

The literature on axiomatic cost sharing with variable demands is almost unanimous in recommending one cost sharing method known as the *Aumann-Shapley pricing* method and computed as follows. If $q = (q_1, ..., q_n)$ is the demand profile, the cost share of good i is

$$x_i(q; C) = \int_0^{q_i} \partial_i C\left(\frac{t}{q_i} \cdot q\right) dt \tag{1}$$

(Note that the above is a revenue, not a price; the literature on Aumann–Shapley pricing usually focuses on the per unit price x_i/q_i.)

[4] To the best of our knowledge, the only exceptions are [49] and [20].

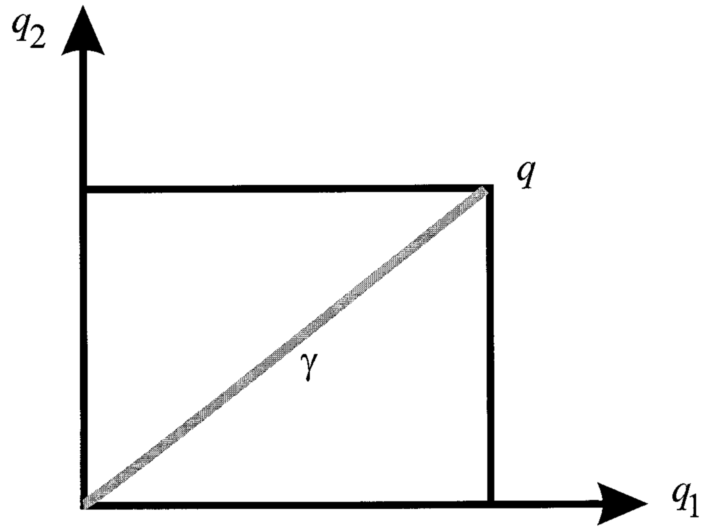


FIG. 1. The Aumann–Shapley method: $x_i = \int_\gamma \partial_i C$.

Here the cost share x_i imputed to agent i is the integral of the partial derivative of C with respect to good i, along the line from the origin to the profile q of individual demands. See Fig. 1.

This method is known to be characterized,[5] within the family of methods meeting additivity and dummy, by the combination of scale invariance and average cost pricing for homogeneous goods. The former requires that a change of the unit in which a particular good is measured should have no effect on cost shares; the latter requires that when the demands q_i enter additively in the cost function (taking the form $C(q_1 + \cdots + q_n)$), cost shares should be proportional to demands. Both axioms are discussed in Section 3.

We challenge this dominant viewpoint in three ways:

(i) we criticize the Aumann–Shapley pricing method because it fails the natural requirement of demand monotonicity: the cost share of a good should not decrease when its demand increases, ceteris paribus.

(ii) we show that the familiar Shapley–Shubik method based on stand alone costs meets the scale invariance and demand monotonicity axioms, and in a certain sense is the only one with these properties.

(iii) we introduce a new axiom, upper bound for homogeneous goods, placing a natural upper bound on the cost share of a good when

[5] All references to the literature are gathered in Section 4.

goods enter additively in the cost function. This axiom points to a third method, serial cost sharing, extending to the present model with heterogeneous goods the method introduced in the homogeneous good model by Moulin and Shenker [32, 33].

To illustrate point (i) we give a numerical example with two goods. The Aumann–Shapley method computes as follows the cost share of agent 1:

$$x_1(q_1, q_2; C) = \int_0^{q_1} \partial_1 C\left(t, t\frac{q_2}{q_1}\right) dt = q_1 \int_0^1 \partial_1 C(tq_1, tq_2)\, dt$$

For the cost function $\tilde{C}(q) = (q_1 \cdot q_2)/(q_1 + q_2)$ this gives

$$x_1(q_1, q_2; \tilde{C}) = \frac{q_1 \cdot q_2^2}{(q_1 + q_2)^2}$$

which is not monotonic in q_1, so that DM fails.

We describe now the two methods mentioned in points (ii) and (iii). The Shapley-Shubik method applies directly the Shapley value formula to the TU-cooperative game defined by the *stand alone costs*. Given a set S of goods (a nonempty subset of $\{1, ..., n\}$), and a profile of demands $q = (q_1, ..., q_n)$, the stand alone cost of coalition S at profile q is

$$c(S; q) = C(q(S)) \qquad \text{where} \quad q(S)_i = q_i \qquad \text{if} \quad i \in S \text{ and zero otherwise.}$$

Then the Shapley-Shubik formula imputes to good i a convex combination of its incremental costs $c(S \cup \{i\}; q) - c(S; q)$ where S spans the entire set of coalitions not containing i.[6]

The serial cost sharing method extends to heterogeneous goods a formula originally introduced in the homogeneous good problem. It is given by an integral formula similar to (1), namely,

$$x_i(q; C) = \int_0^{q_i} \partial_i C((te) \wedge q)\, dt$$

$$\text{where} \quad e = (1, ..., 1) \qquad \text{and} \qquad (a \wedge b)_i = \min(a_i, b_i). \tag{2}$$

The cost share x_i of agent i is the integral of the partial $\partial_i C$ along a path joining 0 to q by raising all coordinates at the same speed and freezing a coordinate once it reaches q_i. See Fig. 2.

[6] The formula is $x_i(q; C) = \sum_{S \subseteq N \setminus \{i\}} (s!\,(n-s-1)!/n!)\, \{c(S \cup \{i\}; q) - c(S; q)\}$.

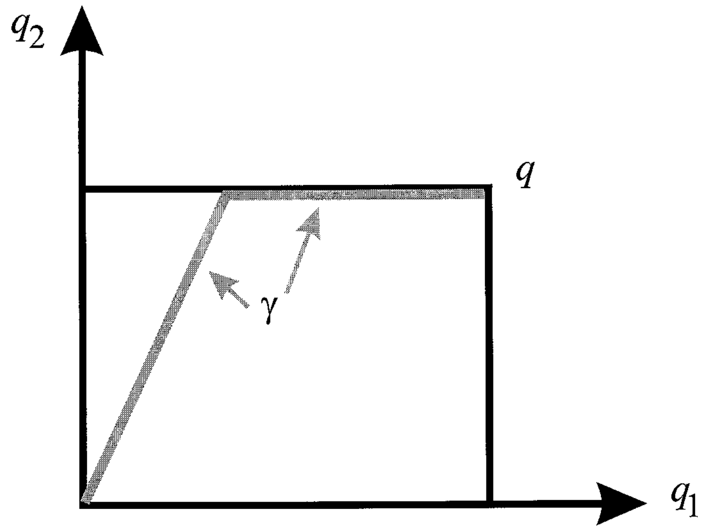


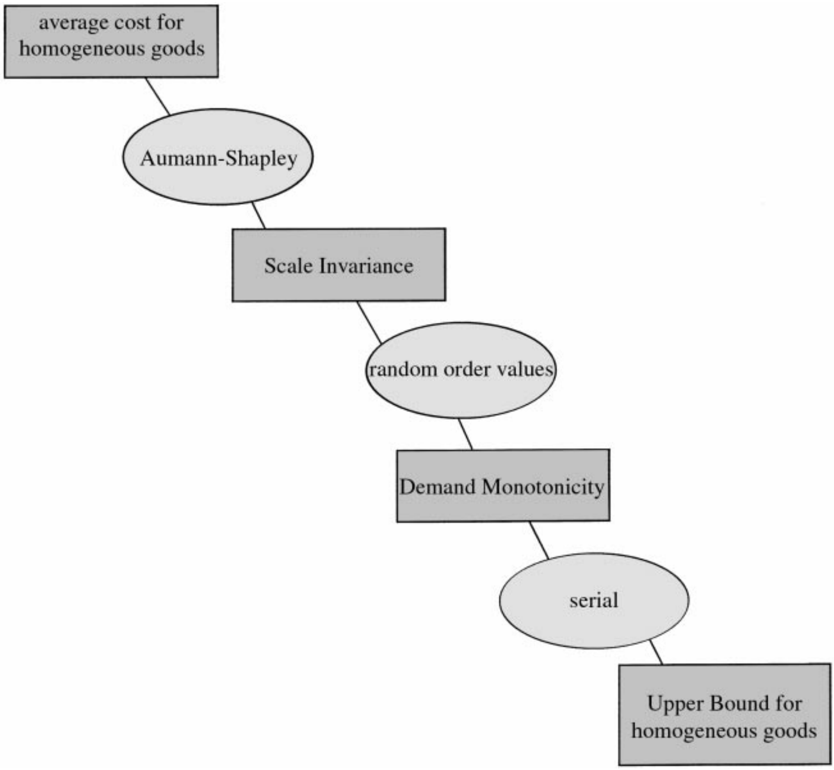
FIG. 2. Serial cost sharing: $x_i = \int_\gamma \partial_i C$.

Within the family of methods meeting additivity and dummy, the logical implications of the four axioms are striking: there are exactly three cost sharing methods meeting more than one axiom (if we count the family of random order values as "one" method); each such method meets exactly two axioms and is characterized by this pair:

- the random order values are characterized by the pair SI *plus* DM; adding a mild symmetry requirement picks the Shapley–Shubik method—Theorem 1 and Corollary in Section 7
- serial cost sharing is characterized by the pair DM *plus* UBH—Theorem 2 in Section 8
- Aumann–Shapley pricing is characterized by the pair SI *plus* ACPH: this is well known and repeated in Section 8.

Finally, the other three pairs from our four axioms, namely ACPH + DM, ACPH + UBH, SI + UBH, are incompatible; it follows that any three of the four axioms are incompatible.

A diagram where axioms are depicted in rectangles and cost sharing methods in ovals visualizes our findings. A pair of axioms not connected by a line are mutually incompatible; only three pairs of axioms are thus connected, each pair characterizing one of the three methods under discussion.



3. INTERPRETING THE FOUR AXIOMS

One interpretation of our results has the familiar impossibility flavor: our four axioms are convincing, at least in certain contexts, and we find it impossible to meet more than two of these at a time. We prefer the "positive" interpretation: beyond the familiar axiomatic foundation of Aumann–Shapley pricing, we offer axiomatic foundations for two other cost sharing methods based on similar axioms. The choice of one of these three methods must depend on the context and this choice is informed by the relevance of the axioms. We discuss successively the four axioms in this spirit.

Demand monotonicity is our first new[7] axiom: when the level of the demand q_i increases while every other demand stays put, the cost share imputed to good i should not decrease. When a different agent i is associated with each good i, as in the chain store example, the overhead

[7] In point of fact, demand monotonicity was introduced first in [30], for the related cost sharing model where each demand comes in indivisible units; see the discussion of this paper in Section 4.

sharing among divisions example, or the delay sharing among computers, demand monotonicity is a compelling incentive requirement. If DM fails an agent can manipulate in a particularly wasteful way: raising artificially her demand in order to decrease her total bill. In a surplus sharing context, failure to meet DM opens the door to "sabotage."[8] Moreover, DM is a minimal equity requirement, imposing a positive correlation between input contribution and output share.[9]

On the other hand, when the demand of a given good is an aggregate of—small—demands by individual agents, as in the telephone pricing example, or, more generally, in the problem of pricing a regulated monopoly, demand monotonicity is no longer compelling. The aggregate demand of a particular good does not correspond to a normatively relevant group of agents (this is especially true if individual agents may demand a positive quantity of several goods), therefore the equity and incentives justifications of the axiom disappear.

Scale invariance is quite convincing when we take seriously the heterogeneity of the goods. The classical example is the "telephone pricing among professors" [5]. We have n classes of phone calls, differentiated by type (local, long distance; peak time, off-peak) and possibly by the telephone set from which the call originates. There is no direct way to compare quantities of the various phone calls (no obvious exchange rate between local and long distance calls) and the cost shares should be independent of the units in which these different types of good are measured, which is precisely the scale invariance requirement (defined in Section 7). Another standard story (similar to the "overhead division" story in Shubik [47]) is a surplus sharing model. Consider an integrated chain store (e.g., grocery stores) with specialized units supplying the various products offered in the stores (say factor i is meat, factor j is produce and so on). Total revenue (net of the cost of distributing the products to the various stores of the chain, and of other overhead costs) $C(q_1, ..., q_n)$ must be divided between the units. Quantities of apples are not intrinsically comparable to quantities of meat, hence the units in which apples are measured (kilos, pounds or bushels) should not matter.[10]

[8] "If rewards were negatively correlated with productivity, the organization would be subject to sabotage," [2].

[9] This is called the "ordinal equity" principle in the social psychology literature, e.g., [16, 11].

[10] In fact, there is no reason to limit ourselves to *linear* changes of unit; maybe even a non-linear redefinition of the scalar measure of any good should not matter. This is desirable when a good has no natural measurement scale invariant to affine transformations; examples include heat, effort, and welfare level. Of the cost sharing methods discussed in this paper, only the random order values are ordinal. Sprumont and Wang [51] show that they are characterized by this "ordinal invariance" property, in combination with additivity and dummy. Seed also a related result, Proposition 5 in [30], in the "discrete" model discussed at the end of Section 4.

In some contexts, on the other hand, scale invariance is not compelling, because the different goods entering the cost or production function are genuinely comparable, and yet they enter nonadditively in the cost or production function.

A simple surplus sharing example is the familiar production cooperative [42, 17] where good i is the effort level provided by worker i. Here different workers hold different jobs in the factory, so that their inputs may not enter symmetrically in the production function (C is not symmetrical in its variables). Yet their respective effort levels are comparable, and this comparison, together with the consideration of the entire production function, determine their final shares of output.

Our next example is an important problem in network design. Consider a group of agents sharing a single network link.[11] Let q_i be player i's (stochastic) transmission rate, i.e., the expected number of packets sent per second. The choice of service protocol, such as first in first out or round robin, determines the probabilistic delays imposed on the players. Let x_i be the average number of player i's packets residing in the queue, which is related to the average queueing delay d_i by Little's law [19]. When the transmissions follow a Poisson process and the service times are I.I.D. exponential with rate a (known as the M/M1 queue) work-conservation implies that for any nonwasteful service protocol $\sum_{i=1}^n x_i = (\sum_{i=1}^n q_i)/(a - \sum_{i=1}^n q_i) = C(q)$. In other words, the choice of a service protocol amounts to the choice of a cost sharing method.[12]

In the queueing model just described, the demand q_i represents the size of the service requested by network user i, whereas x_i is the delay imposed on agent i by the network. A different problem is scheduling: here user i requests to be served before a certain deadline q_i , and the server incurs costs $C(q)$ to meet the deadlines of all the users, so that x_i is the monetary cost share of agent i.

The common feature of these queueing, scheduling or production cooperative stories is that the demands of two different agents are comparable, so that scale invariance is not compelling. Of course, the weaker property requiring that a change in the *common* unit of the goods should have no effect, is compelling. All the methods discussed in this paper are invariant to a rescaling of the common unit.

[11] This general queueing model applies to many other situations, such as information requests from a database or employees sending questions via email to their boss.

[12] There are other constraints, $\sum_{i \in S} x_i \geq C(q_S, 0)$ for all $S \subset \{1, ..., n\}$, that also must be satisfied: [9]. Interestingly, for the M/M/1 queue, all of these are satisfied by the three methods which we will discuss in this paper. In fact Aumann–Shapley corresponds to first in first out queueing, incremental cost sharing (the extremal random order values) to preemptive priority service, and serial cost to fair queueing [45].

The next axiom, ACPH, is a familiar requirement of the literature on Aumann–Shapley pricing: it imposes the form of our cost sharing method on a restricted class of cost functions, namely these cost functions $C(q_1 + \cdots + q_n)$ where the various goods enter additively. Thus the goods can be conceived as one single *homeogeneous* good, because the technology treats the goods as if they were identical. This axiom, as well as the next one, UBH, is distinctly less compelling than properties like DM or SI, that impose a natural requirement to any cost function in the admissible domain.

Average cost pricing for homogeneous goods is compelling when the demand q_i of good i is the aggregate of a number of individual demands that cannot be individually monitored. In the telephone pricing example, assume that only one type of calls (say local calls) are used, so that the demands q_i emanating from different telephone sets enter additively in the cost; if local calls made from different phones were charged differently, users would have an arbitrage opportunity: e.g., they would transfer some of their demand to the cheap phone set. It is well known that average cost pricing for homogeneous goods is the only method immune to manipulations by transfers of demands across agents, by merging several demands into one, or splitting one demand into several and so on.[13]

On the other hand, ACPH is *not* compelling when the agents and their demands are easily monitored. In the chain store example above the suppliers of a certain kind of apples, whose inputs q_i enter additively in the production function, cannot easily collude because they are in different locations and transferring goods is costly. Or take the cost sharing story told by Shubik [47]: a firm has n divisions, each division manager requests q_i units of good i from the central unit and $C(q_1, ..., q_n)$ is total overhead cost to be allocated between the divisions. Even if the goods i, j enter additively in the cost function, the actual quantities consumed in each division are easily monitored so as to prevent collusive transfers; merging and splitting divisions are simply not feasible. And, finally, in the delay-sharing among computers story, the decentralized nature of the network makes collusion impractical.

The discussion of the ACPH axiom revolves around the distinction between two stylized cases. In the first context the demand for good i is an anonymous agregate and the mechanism has no way to know who is behind what fraction of the demand of what good. In the second, good i is impersonated by a specific agent i who consumes the entire amount q_i (and consumes no other good); ACPH is compelling in the former context but much less so in the latter. On the other hand, DM is compelling in the latter context but much less so in the former.

[13] See [3, 7, 28, 33].

The fourth axiom is new. Like ACPH, it places a restriction on the method over the restricted class of cost functions with homogeneous goods. Unlike ACPH, it does not *define* the method over this subclass; moreover the axiom is still meaningful for more general cost functions than homogeneous ones.

Upper bound for homogeneous goods. When goods are homogeneous, i.e., enter additively in the cost function, the axiom requires that no agent should pay more than the cost of producing n times his own demand, where n is the total number of agents. This bound protects the agent with a below average demand from being charged too much: if marginal costs increase, the low demand agent should not be held responsible for the high marginal cost generated by demands larger than his own. Symmetrically, in the surplus sharing interpretation, UBH places a cap on the surplus share awarded to an agent who contributes less than the average contribution.

For instance, with two goods and the cost function $C(q_1, q_2) = (q_1 + q_2)^2$, UBH requires that each agent i be charged no more than $C(q_i, q_i) = 4q_i^2$, irrespective of the demand q_j by agent j. This requirement is not met by average cost pricing, which charges $q_i \cdot (q_1 + q_2)$. In general, if the goods $i = 1, ..., n$ are different but comparable (as in the queuing problems described above), and if they enter symmetrically (but not necessarily additively) in the cost function, the UBH axiom is still meaningful.

In the congestion example (delay sharing among computers, see above) UBH is a natural requirement, because we wish to protect the small users from suffering excessive externalities (i.e., delays) from large users. This reflects a concern for equity, and has incentives implications as well: users are less likely to drop from the queue when they know an upper bound on their own waiting time that does not depend upon the level of demand by other users.

We show in Section 8 that the axiom UBH is not compatible with scale invariance (given that we also require additivity and dummy). In problems where the goods are truly heterogeneous and their units incomparable (such as in the chain store or telephone pricing stories), scale invariance is compelling whereas the UBH requirement must be dropped.

4. RELATION TO THE LITERATURE

Our approach falls squarely in the axiomatic tradition, following Shapley's seminal work in the cooperative game model [43] which inspired much of the social choice and axiomatic bargaining literatures, as well as the microeconomic theory of distributive justice. For a good survey, see [39]. The choice of a cost or surplus sharing method is conceptually analogous to the choice of a constitution behind the “veil of ignorance” [36]. In our model, the mechanism designer can be thought of as a chain store manager setting up rules for allocating joint costs that will apply to every particular store with its particular cost structure. Indeed, the axiomatic analysis hinges on the assumption that cost functions may span a rich domain, in this case any nondecreasing and smooth cost function of n variables. See Section 5.

A striking feature of the axiomatic approach is to be welfare-blind and incentive-blind: preferences of the users of the technology, over output and cost shares, are entirely absent from the model. Compare with the “Bayesian” tradition in mechanism design whereby the designer uses his/her incomplete information about the cost function, as well as about the users characteristics, when choosing a particular cost sharing method. See, e.g., [21]. This latter approach is well suited to analyze the allocation of joint costs, or of whatever resources, on a short term basis when information about (the distribution) of agents’ preferences is well known, and is common knowledge among all participants. Such results are often fragile, depending delicately on the detailed structure of the game’s priors, and thus can be problematic in the public goods arena, or in situations when preferences are not common knowledge, such as on the Internet. In contrast the axiomatic methodology typically leads to (comparatively) simple and robust methods which may be more politically acceptable because they do not rely on any information about preferences. The axiomatic method is the only route toward uncovering general principles of distributive justice. In the spirit of Aristotle’s “proportionality principle,” [16] if offers fair division methods with broad applicability and parsimonious informational requirements.

We now review the literature on axiomatic cost sharing with variable demands. The first paper directly relevant is Shubik [47] who proposes the Shapley-Shubik method discussed above (and of which the corollary to Theorem 1 gives an axiomatic characterization).

The literature on the Aumann–Shapley pricing method starts with the parallel characterizations by [4] and [27]. [24] and [25] discuss nonanonymous versions of the Aumann–Shapley pricing method and offer a characterization of the random order values related to our Theorem 1. The crucial difference is that their strong additivity axiom imposes the requirement that only stand alone costs should matter, whereas our Theorem 1 derives this property. The main point of our paper is to show how various combinations of natural axioms such as DM, SI, and UBH force the method to use information only about a very small subset of the rectangle $[0, q]$, namely a path joining 0 to q or the finite set of vertices. Sprumont and Wang [51] offer an alternative characterization of the random order values related to our Theorem 1: they replace the two requirements DM and SI by a single axiom called ordinality and requiring cost shares to be invariant under a nonlinear rescaling of any good.

Serial cost sharing was introduced in Shenker [44] and its incentives properties analyzed by Moulin and Shenker [32]: more on these properties in the concluding Section 9. More relevant to the current paper is [33], developing an axiomatic comparison of average cost pricing and serial cost sharing when the goods are homogeneous, i.e., on the domain of cost functions taking the form $C(q_1 + \cdots + q_n)$. There the additivity and separability axioms, the latter being a counterpart to dummy in the homogeneous good case, play a central role, as does the upper bound axiom. However scale invariance and demand monotonicity have no counterpart. See Remark 2 at the end of Section 8.

This paper is related to (inspired by) [30], developing a parallel theory of cost sharing with variable demands when goods are heterogeneous and come in indivisible units (so each demand q_i takes integer values). In that "discrete" model, no topological difficulties stand in the way of systematic characterization results. Moulin [30] offers a characterization of serial cost sharing in the discrete model that closely resembles our Theorem 3 (Proposition 4, statement ii); it offers a characterization of the Shapley–Shubik method by imposing an ordinality requirement, later weakened by Sprumont and Wang [52] into a discrete analog of scale invariance. Other related work is [53] and [50], discussing the discrete version of the Aumann–Shapley method: the former paper characterizes this method while the latter criticizes it.

A key step in our analysis is the representation result (Lemma 3) describing the set of cost sharing methods satisfying additivity and dummy. It is inspired by the main result in Moulin [30]. Since this paper was written, a more elegant representation theorem has been given by Haimanko [13]. It shows that the extreme points of this set consist of taking partial derivatives of the cost function along a monotone path: this result is described in Section 6. Wang [55] establishes its counterpart in the discrete model. Finally Friedman [12] exploits the (continuous) representation result to analyze the cost sharing methods satisfying some of the axioms scale invariance, demand monotonicity and consistency.

5. THE MODEL

We fix the set $N = \{1, ..., n\}$ of heterogeneous goods (also interpreted as agents) throughout the paper.

Notations. Given p, q, both in $\mathbb{R}^n$ and such that $p \leqslant q$, the rectangle $[p, q]$ is defined by the inequalities $p \leqslant z \leqslant q$. Given a subset S of N, we write $(q\backslash^S p)$ for the vector with ith coordinate q_i if $i \notin S$ and p_i if $i \in S$. Given $q \in \mathbb{R}^n_+$ we write $q_N = \sum_1^n q_i$ and $q_S = \sum_S q_i$. Given a differentiable real valued function C with domain $\mathbb{R}^n_+$, we denote by $\partial_i C(q)$ its partial derivative at q in the ith coordinate. Given an element q of $\mathbb{R}^n_+$, we denote by $\Gamma(q)$ the vector space of real valued continuously differentiable functions with domain $[0, q]$ such that $C(0) = 0$, and by $\Gamma_+(q)$ the subset of its non-decreasing elements ($p \leqslant p' \Rightarrow C(p) \leqslant C(p')$ for all p, p' in $[0, q]$). The notations $\Gamma(\infty)$, $\Gamma_+(\infty)$ stand for functions with domain $\mathbb{R}^n_+$ (and the same regularity/monotonicity properties). Finally the notations $\Gamma^{-i}(q)$ (resp. $\Gamma^{-i}(\infty)$, $\Gamma^{-i}_+(q)$, ...) stand for the subset of $\Gamma(q)$ (resp. $\Gamma(\infty)$, $\Gamma_+(q)$...) made of functions independent of q_i, in other words:

$$C \in \Gamma^{-i}(q) \Leftrightarrow C \in \Gamma(q) \qquad \text{and} \qquad \{\partial_i C(p) = 0 \text{ for all } p \in [0, q]\}.$$

We now introduce the central concept of this paper: A *cost sharing method* is a mapping x from $\mathbb{R}^n_+ \times \Gamma_+(\infty)$ into $\mathbb{R}^n_+$ associating to each demand profile q and cost function C a vector $x(q; C)$ of cost shares satisfying the budget balance property:

$$\sum_{i=1}^{n} x_i(q; C) = C(q)$$

(note that cost shares are non negative by assumption). The two basic axioms that we maintain throughout are now defined.

Additivity (*ADD*). For all C^1, C^2 in $\Gamma_+(\infty)$ and all $q \in \mathbb{R}^n_+$,

$$x(q; C^1 + C^2) = x(q; C^1) + x(q; C^2).$$

Dummy (*DUM*). For all $C \in \Gamma_+(\infty)$, all $i \in \{1, ..., n\}$,

$$\{C \in \Gamma^{-i}_+(\infty)\} \Rightarrow \{x_i(q; C) = 0 \text{ for all } q \in \mathbb{R}^n_+\}.$$

We denote by $\mathscr{C}_0$ the set of cost sharing methods and by $\mathscr{C}$ the subset defined by the combination of additivity and dummy. Additivity is a decentralizability property: it does not matter whether we compute cost shares on the two separate components C^1 and C^2 of total costs or we apply the formula to the combined cost. Dummy says that a good that can be produced at no cost, no matter how much of the other goods is produced, should not be charged any cost (the assumption $\partial_i C \equiv 0$ means that good i is "free;" we also say that good i is a dummy good). This is the straightforward generalization of the original dummy axiom for cooperative games [43].

6. AN INTEGRAL REPRESENTATION UNDER ADDITIVITY AND DUMMY

In this section, we present an integral representation of $\mathscr{C}$, on which all subsequent arguments (and our four theorems) are based.

Throughout this section we fix a cost sharing method x in $\mathscr{C}$.

LEMMA 1 (Independence of Irrelevant Costs). *Fix $q \in \mathbb{R}^n_+$ and two cost functions C^1, C^2 in $\Gamma_+(\infty)$ that coincide on $[0, q]$. Then $x(q; C^1) = x(q; C^2)$.*

Lemma 1 implies that for each fixed $q \in \mathbb{R}^n_+$, the cost sharing method x defines an additive operator $C \rightarrow x(q; C)$ with domain $\Gamma_+(q)$ (as no confusion may arise we use the same notation).

LEMMA 2. *For every fixed q in $\mathbb{R}^n_+$, the operator $x(q; \cdot)$ extends uniquely to a linear operator on $\Gamma(q)$. The extended operator is denoted $x(q; \cdot)$ as well.*

In Lemma 1 and 2, only the additivity property of x was used. Dummy is essential for the proof of the next result, namely the integral representation.

LEMMA 3. *For all $q \in \mathbb{R}^n_+$ and all i, there exists a nonnegative Radon measure $\mu_i(q)$ on $[0, q]$ such that for all $C \in \Gamma(q)$:*

$$x_i(q; C) = \int_{[0, q]} \partial_i C(p) \, d\mu_i(q)(p). \tag{3}$$

Moreover the projection of $\mu_i(q)$ on the (one-dimensional) interval $[0, q_i]$ is the Lebesgue measure:

$$\begin{aligned} &\textit{for all} \quad a, b, 0 \leqslant a \leqslant b \leqslant q_i, \\ &\textit{set} \quad \Omega = \{ p \in [0, q] \mid a \leqslant p_i \leqslant b \} \\ &\textit{then} \quad \mu_i(q)(\Omega) = b - a. \end{aligned} \tag{4}$$

The proof is an application of the Riesz representation theorem. See Appendix 1.

Note that, as stated, Lemma 3 is not a complete characterization of $\mathscr{C}$, because it does not take into account budget balance. The complete representation theorem is significantly more complicated to state and is not required for the proofs of this paper; it is presented in Appendix 1. The conditions that the measures must satisfy to give a valid method in $\mathscr{C}$ are quite complex, consisting of 2^n restrictions (see (14) in Appendix 1). Since we first completed this analysis a more elegant representation has been discovered. It relies on the family of "path generated" methods, that compute the cost share of good i as the integral of the ith partial derivative along a given path.

This representation is based on the fact that the set $\mathscr{C}$ is convex and that the methods generated by a path are the extreme points of this set. As we noted in Section 2 the Aumann–Shapley method ((1)) is generated by a path which is the diagonal from 0 to q (Fig. 1), while serial cost sharing ((2)) is also generated by a path (Fig. 2). Next consider the following *incremental cost sharing method* among two agents

$$x_1(q;\,C)=C(q_1,0);\qquad x_2(q;\,C)=C(q_1,q_2)-C(q_1,0)$$

(incremental methods are defined for any N and characterized in Section 7). This method is generated by the path from 0 to x along the lower edge of the rectangle $[0,x]$: this path is denoted $\gamma_{1,2}$ on Fig. 3. The symmetric incremental method

$$x_1(q;\,C)=C(q_1,q_2)-C(0,q_2);\qquad x_2(q;\,C)=C(0,q_2)$$

is similarly generated by the path (denoted $\gamma_{2,1}$ on Fig. 3) along the upper edge of this rectangle. Then the Shapley–Shubik method is simply the average of these two methods: it is *not* generated by the average of these two paths (which is a piecewise linear path depicted on Fig. 3), or by any other path.

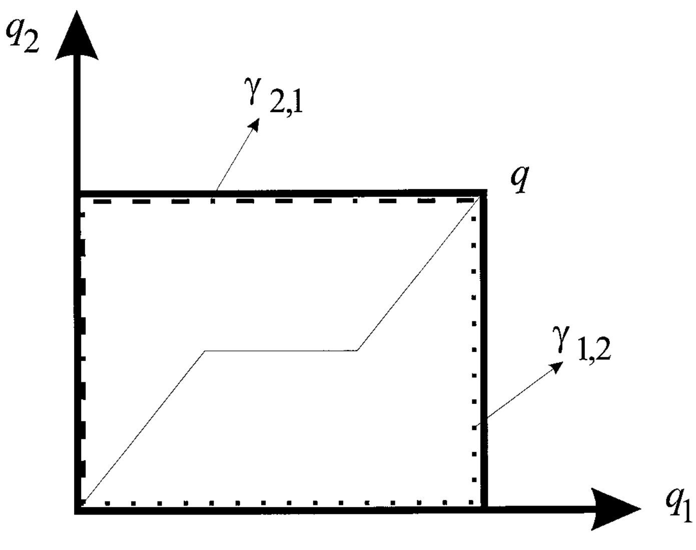


FIG. 3. The two incremental methods when $N=\{1,2\}$.

It turns out that all cost sharing methods in $\mathscr{C}$ can be constructed in a similar manner, as the convex combination of path methods. First note that for any path $\gamma(t; q)$ which is continuous and nondecreasing in t, with $\gamma(0; q) = 0$ and $\gamma(1; q) = q$, the related function

$$x_i^\gamma(q; C) = \int_0^1 \partial_i C(\gamma(t; q))\, d\gamma(t; q)$$

is a cost sharing method in $\mathscr{C}$. These methods form the extreme points of the set $\mathscr{C}$, as shown by Haimanko [13].

THEOREM. *$x(q; \cdot)$ is an extreme point of the set $\mathscr{C}$ (for fixed q) if and only if there exists some path $\gamma(q; \cdot)$ such that $x(q; \cdot) = x^\gamma(q; \cdot)$.*

As discussed in Friedman [12] using Lemma 3 and Choqet's theorem [35], we can use this result to get a true representation theorem.

COROLLARY. *If $x(q; \cdot)$ is a csm in $\mathscr{C}$, there exists a measure v on $\Gamma(q)$ such that*

$$x(q; \cdot) = \int_{\Gamma(q)} x^\gamma(q; \cdot)\, dv(\gamma),$$

where $\Gamma(q)$ is the set of all paths from 0 to q.

We refer the reader to [12] for the details, such as the definition of the measure spaces and various topologies. That paper uses the representation by path methods to provide simple representations for various subsets of $\mathscr{C}$, such as those satisfying scale invariance, or demand monotonicity, or consistency.

Remark 1. By formula (3), all cost sharing methods in $\mathscr{C}$ satisfy also the marginality property [58]: the cost share imputed to good i depends only upon the marginal cost function with respect to good i. In the standard cooperative game context (where each demand q_i equals zero or one), [57] offers a characterization of the Shapley value where marginality replaces both additivity and dummy. (This result generalizes to a characterization of the random order values: [18]) In the variable demand context of this paper, marginality is not enough to replace additivity: it is easy to construct non additive methods meeting marginality.

Remark 2. Lemma 3 implies a robustness of the cost shares: if two cost functions are close (in the C^1 metric) then the associated cost shares will also be close. Thus in this metric an approximate knowledge of the cost function allows one to compute approximate cost shares.

7. SCALE INVARIANCE, DEMAND MONOTONICITY, AND THE RANDOM ORDER VALUES

We define two important axioms for cost sharing methods. The following notation will be useful to state SI: for any positive number λ, the mapping τ^i_λ is the λ-rescaling of the ith coordinate

$$\tau^i_\lambda(q) = (q \backslash^i \lambda q_i) \qquad \text{for all} \quad q \in \mathbb{R}^n_+, \text{ all } i, \text{ all } \lambda > 0$$

(where $(q \backslash^i q'_i)$ equals q except that its i-coordinate is q'_i).

Given a cost function C it is transformed into the function $\tau^i_\lambda[C]$ by the said λ-rescaling

$$\tau^i_\lambda[C](q) = C(\tau^i_{1/\lambda}(q)).$$

Scale invariance. For every i, every cost function $C \in \Gamma_+(\infty)$, every $q \in \mathbb{R}^n_+$ and every $\lambda > 0$:

$$x(q; C) = x(\tau^i_\lambda(q); \tau^i_\lambda[C]).$$

Demand monotonicity. For every i, every cost function $C \in \Gamma_+(\infty)$ and every $q \in \mathbb{R}^n_+$:

$$\text{for any } q'_i \quad \{q_i \leqslant q'_i\} \Rightarrow \{x_i(q; C) \leqslant x_i((q \backslash^i q'_i); C)\}$$

The SI and DM axioms are discussed in Section 3.

We showed in Section 3 that the Aumann–Shapley pricing method is not demand monotonic. Here we check, also by means of a numerical example, that serial cost sharing fails scale invariance. With two goods and demands q_1, q_2 such that $q_1 \leqslant q_2$, serial cost sharing yields the cost share:

$$x_1(q_1, q_2; C) = \int_0^{q_1} \partial_1 C(\tau, \tau)\, d\tau.$$

Consider the cost function $C(q) = (q_1 + q_2)^2$, then double the scale of good 1 so that the "new" demand $\tilde{q}_1$ is worth $\tilde{q}_1 = (1/2)\, q_1$ if q_1 is the "old" one. The new cost function is $\tilde{C}(\tilde{q}_1, q_2) = (2\tilde{q}_1 + q_2)^2$ and we have

$$x_1(1, 1; C) = 2$$

$$x_1(\tfrac{1}{2}, 1; \tilde{C}) = \int_0^{1/2} 4 \cdot (2\tau + \tau)\, d\tau = \tfrac{3}{2}$$

in violation of scale invariance.

If serial cost sharing fails SI, it is easy to see that it is invariant upon a *simultaneous* rescaling of *all* goods.

We now define the *random order values* [56]. Fix an ordering σ of $\{1, 2, ..., n\}$ and for every agent i, denote by $S(\sigma; i)$ the set of agents preceding i in this ordering, i.e., $j \in S(\sigma; i)$ if $\sigma(j) < \sigma(i)$. Recall the notation $c(S; q) = C(q(S))$ where $q(S)_i = q_i$ if $i \in S$ and zero otherwise. The *incremental cost sharing method* associated with σ is defined by

$$x_i(q; C) = c(S(\sigma; i) \cup \{i\}; q) - c(S(\sigma; i); q). \tag{5}$$

A *random order value* is an arbitrary convex combination of incremental c.s.m.s. (the convex weights being independent of both q and C). A random order value is an element of $\mathscr{C}$. Naturally, the Shapley–Shubik method is the arithmetic average of the $n!$ incremental methods. Figure 3 shows the paths generating the two incremental methods when $N = \{1, 2\}$.

The last ingredient of Theorem 1 is a technical continuity requirement for null demands:

Continuity at zero. For every i, every $C \in \Gamma_+(\infty)$ and every $q \in \mathbb{R}^n_+$:

$$\lim_{q'_i > 0,\, q'_i \to 0} x((q \backslash^i q'_i); C) = x(q \backslash^i 0), C).$$

THEOREM 1. *Given N and a cost sharing method x in* $\mathscr{C}$ (*hence satisfying additivity and dummy*) *the two following statements are equivalent*:

(i) *x is a random order value*;

(ii) *x satisfies scale invariance, demand monotonicity and continuity at zero.*

COROLLARY. *The Shapley–Shubik method is characterized by the combination of the five axioms ADD, DUM, SI, DM, Continuity at zero, and the following symmetry property*:

Symmetry. If $C \in \Gamma_+(\infty)$ is a symmetrical function of the n variables q_i then $x(e; C) = (1/n)\, C(e)$ where $e = (1, 1, ..., 1)$.

Remark 3. All random order values are continuous in q everywhere. In fact any method in $\mathscr{C}$ meeting SI must be continuous in q for any q such that $q_i > 0$ for all i (this follows easily from Lemma 3). However, if we drop the requirement of Continuity at zero, new cost sharing methods emerge: they are described in Remark 5 after the proof of Theorem 1 in Appendix 2.

8. UPPER BOUND AND SERIAL COST SHARING

In this section we pay special attention to the important subspace of Γ_+ containing all cost functions of the form $C(q) = C_0(\sum_{i=1}^n q_i)$. For such technologies, the quantities demanded of the various goods enter additively in the cost computation, therefore we may view them as a single homogeneous good (e.g., blue cars and red cars). We denote by $\Gamma^h(q)$ (resp. $\Gamma^h_+(q)$) the subspace of $\Gamma(q)$ (resp. the cone of $\Gamma_+(q)$) containing all cost functions of the above form. We call such cost functions *homogeneous*.

For our second characterization result, we need a slightly more general family of cost functions, for which all goods are either "dummy goods" or enter additively in the argument of the function.

Notation. We denote by Γ^{h*} (resp. Γ^{h*}_+) the set of all cost functions C in Γ^h such that all nondummy goods are homogeneous: there exists $S \subseteq N$ and C_0, a continuously differentiable real valued function on $\mathbb{R}$ with $C_0(0) = 0$, such that $C(q) = C_0(\sum_{i \in S} q_i)$ for all $q \in \mathbb{R}^n_+$ (resp. the function C_0 is nondecreasing as well).

The next result hinges on the upper bound axiom, placing a cap on the cost share of all agents when goods are homogeneous (or dummies).

Upper bound for homogeneous goods (UBH). If C is in Γ^{h*}_+ and $e = (1, ..., 1)$ then for all $i \in N$ and all $q \in \mathbb{R}^N_+ : x_i(q; C) \leqslant C(q_i e)$.

This says that the cost share of agent i cannot exceed the cost of producing k times his own demand q_i, where k is the number of nondummy goods. As discussed in Section 3, UBH protects the below average agent, in the cost sharing context; in the surplus sharing context, it protects the agent with a high (above average) contribution, by limiting the surplus share of the low contribution agents.

THEOREM 2. *Given N and a cost sharing method x in $\mathscr{C}$ (hence satisfying additivity and dummy) the two following statements are equivalent*:

(i) *x is serial cost sharing*:

$$x_i(q; C) = \int_0^{q_i} \partial_i C((te) \wedge q)\, dt$$

$$\text{where} \quad e = (1, ..., 1) \qquad \text{and} \qquad (a \wedge b)_i = \min(a_i, b_i) \qquad (2)$$

(ii) *x satisfies demand monotonicity and upper bound for homogeneous goods.*

Note that for the serial cost sharing method the upper bound $x_i(q; C) \leqslant C(q_i e)$ holds for any cost function in $\Gamma_+(\infty)$. Indeed, for all t such that $0 \leqslant t \leqslant q_i$, we have

$$\partial_i C((te) \wedge q) \leqslant \frac{d}{dt} [C((te) \wedge q)].$$

Hence, by integrating between 0 and q_i,

$$x_i(q; C) \leqslant C((q_i e) \wedge q) \leqslant C(q_i e).$$

This upper bound property has no clear normative meaning for a general cost function in $\Gamma_+(\infty)$, even if the various goods are expressed in a common unit. However, if C is a symmetrical function of the n goods, then comparing the quantities of these goods is meaningful and the inequality $x_i \leqslant C(q_i e)$ retains its appeal.

When all goods are homogeneous, the formula (2) yields the familiar serial cost sharing method, discussed in [32, 33, 44]. Fix a cost function C in $\Gamma^h(\infty)$ and a profile of demands q. Up to relabeling the coordinates, assume $q_1 \leqslant q_2 \leqslant \cdots \leqslant q_n$ and define the auxiliary quantities $q^i = (n-i+1)\, q_i + \sum_{j=1}^{i-1} q_j$ for $i = 1, ..., n$. The serial cost shares are now

$$x_i(q; C) = \frac{C_0(q^i)}{n-i+1} - \sum_{j=1}^{i-1} \frac{C_0(q^j)}{(n-j+1)(n-j)} \qquad \text{where} \quad C(q) = C_0(\Sigma q_i). \quad (6)$$

We give now an alternative characterization of serial cost sharing (for general heterogeneous cost functions) that follows at once from Theorem 2.

Corollary to Theorem 2. *Given N and a cost sharing method x in $\mathscr{C}$, the two following statements are equivalent*: (i) *x is serial cost sharing* (*formula* (2)); (ii) *x satisfies demand monotonicity and equals serial cost sharing* (*formula* (6)) *for homogeneous cost functions.*

The corollary follows from Theorem 2 upon noticing that if the method x equals serial cost sharing ((6)) for homogeneous goods, then it meets UBH.

Remark 4. Theorem 2 is reminiscent of Theorem 3 in Moulin and Shenker [33], offering a characterization of serial cost sharing in the space of homogeneous cost functions by means of three axioms: additivity, upper bound and a weak form of the familiar consistency axiom. The first two axioms are analogs of the ones used in this paper, but the latter is not. Moulin and Shenker [34, Lemma 4] offer yet another characterization of serial cost sharing in the space of homogeneous cost functions by means of three axioms: additivity, upper bound, and the distributivity axiom, requiring that the computation of cost shares commutes with the composition of (one input, one output) cost functions—just like additivity says that the computation of cost shares commutes with the addition of cost functions.

We conclude this section by repeating the classical characterization of Aumann–Shapley pricing and stating the remaining incompatibilities between our four axioms.

Average Cost Pricing for Homogeneous Goods (*ACPH*)

If C is in Γ^h_+, we have $x_i(q; C) = q_i/q_N \cdot C(q)$.

THEOREM 3 [4, 27]. *Given* N *and a cost sharing method* x *in* $\mathscr{C}$, *the following statements are equivalent*:

(i) x *is Aumann–Shapley pricing*: *formula* (1)

(ii) x *satisfies scale invariance and average cost pricing for homogeneous goods.*

Note that the dummy axiom is redundant in the above characterization.

LEMMA 4. *There is no method in* $\mathscr{C}$ *satisfying any one of the following pairs of axioms*:

(i) *Average cost pricing for homogeneous goods and demand monotonicity.*

(ii) *Average cost pricing for homogeneous goods and upper bound for homogeneous goods.*

(iii) *Scale invariance and upper bound for homogeneous goods.*

In fact, scale invariant cost sharing methods do not satisfy *any* upper bound of the form $x_i(q; C) \leqslant \alpha_i(q_i; C)$ where the upper bound is finite and does not depend on q_{-i}. This claim is established in the proof of Lemma 4.

9. CONCLUDING COMMENTS

(a) A variant of the model of this paper uses a fixed, finite, capacity Q_i for each good i. In the variant, the choice of the cost sharing method may depend upon the capacity profile Q. In this fashion, we may adapt the serial formula (6) into a scale invariant (and demand monotonic) method, namely,

$$x_i(q; C) = \int_0^{q_i} \partial_i C\left(\frac{t}{Q_i} e \wedge \frac{q}{Q}\right) dt$$

with the notation $(q/Q)_j = q_j/Q_j$. Naturally, the fact that cost shares at q do depend upon the capacity profile is not very appealing, especially in view of Lemma 1 (easily generalized). If we introduce an axiom of capacity independence (requiring that a change of Q_i be irrelevant as long as it remains above the given demand) then we are back to the current model and results.

(b) The choice of the domain $\mathscr{C}$ of cost functions is an important aspect of our results. Of particular importance is the fact that $\mathscr{C}$ contains cost functions with increasing or with decreasing marginal costs (both $\partial_{ij} C > 0$ or $\partial_{ij} C < 0$ are possible). If we restrict the domain by imposing, for instance, the requirement $\partial_{ij} C \geqslant 0$ for all i, j, then our Theorem 1 is no longer true: [37] shows that on this domain the Aumann–Shapley pricing rule is demand monotonic.

The question of adapting classical axiomatic cost sharing to restricted domains such as the one just mentioned is wholly open. Note that the restriction to the domain where $\partial_{ij} C$ is nonnegative everywhere is natural from the point of view of incentives. On this domain, serial cost sharing (and more generally any method generated by a path independent of the demand q) is incentive compatible in the following sense: its associated demand game where agents choose q_i strategically has an essentially unique Nash equilibrium which is the only rationalizable outcome for all profiles of convex preferences. The converse is also true, at least in the subdomain of homogeneous goods [32] and in the discrete model where goods come in indivisible units [31]. See also [46].

(c) The familiar Theorem 3 and the corollary to Theorem 2 (and the comments following it) suggest to investigate the following general "extension issue." Given an additive cost sharing method over $\Gamma^h_+(\infty)$, there are, in general, several possible ways to extend it into a method in $\mathscr{C}$ (a cost sharing method over $\Gamma_+(\infty)$ meeting additivity and dummy). The extension problem becomes interesting when we require additional properties such as scale invariance or demand monotonicity. We showed that some methods for homogeneous goods (e.g., average cost pricing) can be extended to heterogeneous goods so as to meet scale invariance whereas some other methods (e.g., serial cost sharing) cannot. The problem of characterizing the whole set of methods for homogeneous goods that can be extended while meeting additional properties such as scale invariance or demand monotonicity is the subject of work in progress by the authors, based on the representation theorem discussed in Section 6.

(d) Both Theorems 2 and 3 rely on a "restricted domain" axiom: whenever goods are homogeneous, the method is entirely specified by the axiom (see ACPH) or it satisfies a specific requirement that is normatively meaningful only on this restricted domain (see UBH). This is an unappealing feature of these two axioms; by contrast, Theorem 1 relies on a pair of "full domain" axioms.[14] It is an open question whether either one of the Aumann–Shapley and serial methods can be characterized in $\mathscr{C}$ by full domain axioms.

[14] Of course, dummy is a restricted domain axiom, but one that is hardly objectionable.

APPENDIX 1

Proof of Lemmas 1, 2, *and* 3 *and Representation Theorem*

Proof of Lemma 1. Fix q, C^1, C^2 as above. We claim that we may choose a function D in $\Gamma_+(\infty)$ with the following properties:

$$\text{for all} \quad p \in [0, q] : D(p) = C^1(p) = C^2(p)$$

$$\text{for all} \quad p \in \mathbb{R}^n_+, \text{ all } i\text{: } \partial_i D(p) \geqslant \max\{\partial_i C^1(p), \partial_i C^2(p)\}.$$

(We omit the easy proof of the claim.) Then the functions $D - C^1$ and $D - C^2$ are both in $\Gamma_+(\infty)$ and, by budget balance, we have

$$x(q; D - C^\varepsilon) = 0 \qquad \text{for} \quad \varepsilon = 1, 2.$$

Now additivity yields:

$$x(q; C^1) + x(q; D - C^1) = x(q; C^2) + x(q; D - C^2). \qquad \text{Q.E.D.}$$

Proof of Lemma 2. The operator $x(q; \cdot)$ being additive on $\Gamma_+(q)$ and uniformly bounded below must be linear with respect to positive scalars (by a standard application of Cauchy's theorem: see [1]). That is to say the equality

$$x(q; a^1 C^1 + a^2 C^2) = a^1 x(q; C^1) + a^2 x(q; C^2)$$

must hold for any C^ε in $\Gamma_+(q)$ and any non negative numbers a^ε, $\varepsilon = 1, 2$.

Next we note that every function C in $\Gamma(q)$ is the difference between two elements of $\Gamma_+(q)$. Indeed for a scalar λ large enough, the function $C(p) + \lambda \cdot p_N$ is monotonically increasing, hence the identity

$$C = (C + \lambda \cdot D) - (\lambda \cdot D) \qquad \text{where } D \text{ is the function} \quad D(p) = p_N$$

proves the claim. From the equality

$$\Gamma(q) = \Gamma_+(q) - \Gamma_-(q)$$

it follows that the operator $x(q; \cdot)$ has a unique additive extension to $\Gamma(q)$. From the linearity of $x(q; \cdot)$ with respect to positive linear combinations, it follows that the extended operator is linear (we omit the straightforward argument). Q.E.D.

Proof of Lemma 3. First we state a simple fact. Given arbitrary $q \in \mathbb{R}^n_+$, $i \in N$, and $C \in \Gamma(q)$, there exists a function B in $\Gamma_+(q)$ such that

$$B \in \Gamma_+^{-i}(q) \qquad \text{and} \qquad \{\text{for all } j \neq i : \partial_j(C + B) \geqslant 0\} \tag{7}$$

(i.e., B is independent of q_i and $C+B$ is nondecreasing in q_j for all $j \neq i$). Note that B can even be chosen linear.

Turning to the proof of Lemma 3, we fix a cost sharing method x satisfying additivity and dummy.

Step 1. The cost sharing method x satisfies the following marginal contribution upper bound:

$$\text{for all } i, \text{ all } q \in \mathbb{R}^n_+, \text{ all } C \in \Gamma_+(q) : x_i(q; C) \leqslant q_i \max_{p \in [0, q]} \{\partial_i C(p)\}.$$

Fix i, q and C in $\Gamma_+(q)$, and set $\lambda = \max_{[0,q]}\{\partial_i C\}$. Define the function B^1 with domain $[0, q]$ as follows

$$B^1(p) = \lambda \cdot p_i - (C(p) - C(p \backslash^i 0)) \qquad \text{all} \quad p \in [0, q]$$

and note that B^1 is in $\Gamma(q)$ because C is. Moreover $\partial_i B^1$ is nonnegative by definition of λ. Now we choose a function B^2 in $\Gamma^{-i}_+(q)$ satisfying (7) where B^1 replaces C, and we note that $B = B^1 + B^2$ is nondecreasing in all variables. Thus we have

$$B \in \Gamma_+(q) \qquad \text{and} \qquad \{(C+B)(p) = \lambda \cdot p_i + C(p \backslash^i 0) + B^2(p \backslash^i 0) \text{ for all } p \in [0, q]\}.$$

If D is the function $D(p) = \lambda p_i$, repeated applications of DUM (to all agents but i) gives $x_i(q; D) = \lambda q_i$. Therefore, applying DUM one more time and ADD we get

$$x_i(q; C) + x_i(q; B) = x_i(q; C+B) = \lambda \cdot q_i.$$

As B is in $\Gamma_+(q)$, we have $x_i(q; B) \geqslant 0$ hence the desired inequality $x_i(q; C) \leqslant \lambda \cdot q_i$ follows.

Step 2. The marginal contribution upper bound can be extended to $\Gamma(q)$ as follows:

$$\text{for all } i, \text{ all } q \in \mathbb{R}^n_+, \text{ all } C \in \Gamma(q) : |x_i(q; C)| \leqslant 2q_i \cdot \max_{[0,q]} |\partial_i C(p)|. \tag{8}$$

Fix i, q, and C in $\Gamma(q)$. First we construct C^1, C^2 both in $\Gamma_+(q)$ such that

$$C = C^1 - C^2 \qquad \text{and} \qquad \max_{[0,q]} \{\partial_i C^\varepsilon\} \leqslant 2 \cdot \max_{[0,q]} |\partial_i C|, \qquad \text{for} \quad \varepsilon = 1, 2. \tag{9}$$

We set $\lambda = \max_{[0,q]} |\partial_i C|$ and we choose a function B in $\Gamma^{-i}_+(q)$ satisfying (7) for the function C. Then we define

$$C^1(p) = \lambda \cdot p_i + C(p) + B(p) \qquad \text{for all} \quad p \in [0, q]$$
$$C^2(p) = \lambda \cdot p_i + B(p).$$

By our choice of λ, C^1 is nondecreasing in p_i, hence by construction of B, C^ε is in $\Gamma_+(q)$ for $\varepsilon = 1, 2$. Moreover

$$\partial_i C^1(p) = \lambda + \partial_i C(p) \leqslant 2\lambda; \qquad \partial_i C^2(p) = \lambda$$

completing the proof of property (9). Now we compute

$$\begin{aligned} x_1(q; C) &= x_1(q; C^1) - x_1(q, C^2) \\ &\leqslant x_1(q, C^1) \leqslant q_1 \cdot \max_{[0,q]} \{\partial_1 C^1(p)\} \leqslant 2q_1 \cdot \max_{[0,q]} |\partial_1 C(p)| \end{aligned}$$

and similarly

$$x_1(q, C) \geqslant -x_1(q, C^2) \leqslant -q_1 \cdot \max_{[0,q]} \{\partial_1 C^2(p)\} \leqslant -2q_1 \cdot \max_{[0,q]} |\partial_1 C(p)|$$

and the proof of Step 2 is now complete.

Step 3 (Applying Riesz's theorem). Partial differentiation with respect to the ith coordinate is a linear operator on $\Gamma(q)$. We denote its range by $\nabla(q)$; thus $\nabla(q)$ is a linear subspace of $\Lambda(q)$, the space of continuous functions on $[0, q]$. Observe that $\nabla(q)$ contains all continuously differentiable functions on $[0, q]$ and that the image of $\Gamma_+(q)$ contains all non negative and continuously differentiable functions on $[0, q]$.

(To check the latter claim—the former one is obvious—pick such a function F and define

$$C(p) = \int_0^{p_i} F(p \backslash t_i)\, dt_i.$$

Clearly C is continuously differentiable and nondecreasing in p_i. Adding to C a function of B satisfying (7) we get $D = C + B$, an element of $\Gamma_+(q)$ of which the ith derivative equals F).

We consider now the linear form $C \to x_i(q; C)$ (for a fixed choice of i and C) with domain $\Gamma(q)$. By (8) it is zero on $\Gamma^{-i}(q)$, the kernel of ∂_i in $\Gamma(q)$, hence it decomposes by means of a linear form φ with domain $\nabla(q)$:

$$\text{for all} \quad C \in \Gamma(q) : x_i(q; C) = \varphi(\partial_i C). \tag{10}$$

By (8) again, $\varphi(F)$ is continuous when $\nabla(q)$ is endowed with the L_∞ norm. By the Hahn–Banach theorem (see e.g., [10]), φ can be extended into a continuous linear form on $\Lambda(q)$, endowed with the L_∞ norm (i.e., the uniform convergence topology). By the Riesz representation theorem (see, e.g., [40]), there exists a Radon measure μ on $[0, q]$ such that

$$\text{for all} \quad F \in L(q) : \varphi(F) = \int_{[0,q]} F(p)\, d\mu(p).$$

Combining the above with equation (10) yields the desired formula (3).

Step 4. Proving the nonnegativity of μ and the normalization property (4) (the index i and vector q remain fixed).

If F is nonnegative and continuously differentiable in $[0, q]$, we noted in Step 3 that F is the image (by ∂_i) of a function C in $\Gamma_+(q)$. Therefore $\varphi(F)$ is nonnegative. Now every nonnegative continuous function is the limit (for the L_∞ norm) of continuously differentiable and non negative functions, hence μ is a positive measure.

To check property (4) consider a function C in $\Gamma_+(q)$ and independent of all variables but p_i. By dummy and budget balance, we have

$$C(q_i) = x_i(q; C) = \int_{[0, q]} C'(p_1)\, d\mu(p).$$

The desired conclusion follows by applying the above equality to a sequence of functions C approximating the (nondifferentiable) function

$$C_{a, b}(p_1) = \min\{b - a, \max\{p_1 - a, 0\}\}$$

(i.e., the function with derivative one between a and b and zero elsewhere). Q.E.D.

Representation Theorem

Given q, an agent i, and a measure μ on $[0, q]$, we define its ith semi projected density as follows:

$$\lambda_i(p) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \mu_i([p_i, p_i + \varepsilon] \times [p_{-i}, q_{-i}]).$$

LEMMA. *If $\mu = \mu_i(q)$ is a measure associated with a cost sharing method as in Lemma* 3, *then its ith semi projected density λ_i is defined and satisfies $\lambda_i(p) \leqslant 1$ (Lebesgue) almost everywhere on $[0, q]$.*

Proof. For all p_{-i}, the measure $\hat{\mu}_i$ on $[0, q_i]$ defined by

$$\hat{\mu}_i([a, b]) = \mu_i([a, b] \times [p_{-i}, q_{-i}])$$

is well defined and, by property (4), absolutely continuous with respect to (strictly dominated by) the Lebesgue measure. Now $\lambda_i(p)$, $0 \leqslant p_i \leqslant q_i$, is the density of $\hat{\mu}_i$, hence it is defined almost everywhere. Q.E.D.

LEMMA. *Fix q, and for each i a positive measure $\mu_i = \mu_i(q)$ as in the statement of Lemma* 3 *(i.e., satisfying* (4)*). Then formula* (3) *defines a cost sharing method in $\mathscr{C}$ if and only if we have for all $S \subseteq N$*

$$\sum_{i \in S} \lambda_i(p) = 1 \qquad \textit{a.e. on } [0, q(S)], \tag{11}$$

where $q(S) = (q \backslash^{N \backslash S} 0)$ *and where a.e. is relative to the* $|S|$ *dimensional Lebesgue measure on* $[0, q(S)]$.

Sketch of the Proof

Step 1 ("if"). We assume (11) and check that formula (3) defines a cost sharing method satisfying budget balance (additivity and dummy are obvious). If the cost function C is sufficiently differentiable and $\partial_S C$ is the partial derivative $\partial_{i_1} \cdots \partial_{i_S} C$, where $S = \{i_1, ..., i_S\}$ then we have the identity

$$C(p) = \sum_{S \subseteq N} \int_{[0, p(S)]} \partial_S C(r)\, dr, \tag{12}$$

where "*dr*" is the $|S|$ dimensional Lebesgue measure (this convention is maintained throughout).

Similarly, if the function f is sufficiently differentiable, repeated integration by parts yields the formula

$$\int_{[0, q]} f(p)\, d\mu_i(p) = \sum_{S \subseteq N} \int_{[0, q(S)]} \partial_{S \backslash i} f(p)\, \lambda_i(p)\, dp.$$

Applying the above formula to $f = \partial_i C$ (and invoking (3)) yields

$$x_i(q; C) = \sum_{S \subseteq N} \int_{[0, q(S)]} \partial_S C(p)\, \lambda_i(p)\, dp. \tag{13}$$

Therefore

$$\sum_N x_i(q; C) = \sum_{S \subseteq N} \int_{[0, q(S)]} \partial_S C(p) \cdot \left(\sum_{i \in S} \lambda_i(p) \right) dp$$

and budget balance follows at once from (11) and (12). This proves the claim for all sufficiently differentiable functions C. A straightforward limiting argument extends it to $\Gamma(q)$.

Step 2 ("only if"). Given a family of positive measures μ_i satisfying (4), we define cost shares $x_i(q; C)$ (for fixed q but variable C) by formulas (3). Assuming the budget balance property (for all C), we show that property (11) must be true.

The proof is by contradiction. Let S be a coalition such that (11) does not hold. There must exist an interval $[p^-, p^+]$ in $[0, q(S)]$, with $0 < p_i^- < p_i^+$ for all $i \in S$, $p_i^- = p_i^+ = 0$ for $i \in N \backslash S$ and such that

$$\sum_{i \in S} \int_{[p^-, p^+]} \lambda_i(p)\, dp \neq \prod_{i \in S} (p_i^+ - p_i^-). \tag{14}$$

Consider the characteristic function f_∞ of $[p^-, p^+]$($f_\infty(p) = 1$ if $p \in [p^-, p^+]$ and 0 otherwise). We pick a sequence f_k of smooth functions on $[0, q(S)]$ such that

- $f_k(p) = 0$ if $p_i = 0$ for some $i \in S$
- $\lim_{k \to \infty} f_k(p) = f_\infty(p)$ for all $p \in [0, q]$.

Define the cost functions

$$C_k(p) = \int_{[0, p(S)]} f_k(r)\, dr \qquad k = 1, ..., +\infty. \tag{15}$$

For all $i \in N \backslash S$, we have $x_i(q; C_k) = 0$ all $k = 1, ..., +\infty$ (this follows from formula (3) and $\partial_i C_k = 0$. Next we pick $i \in S$; because C_k is smooth whenever $k < +\infty$, we can use formula (13) to compute $x_i(q; C_k)$. By construction of C_k all terms in the summation on the right hand side of (13) are zero except the S term; hence

$$x_i(q; C_k) = \int_{[0, q(S)]} f_k(p)\, \lambda_i(p)\, dp \qquad \text{for all} \quad k < +\infty.$$

By assumption x is budget balanced; therefore,

$$C_k(q) = \sum_{i \in N} x_i(q; C_k) = \sum_{i \in S} \int_{[0, q(S)]} f_k(p) \cdot \lambda_i(p)\, dp.$$

Letting k go to infinity, we get

$$C(q) = \sum_{i \in S} \int_{[p^-, p^+]} \lambda_i(p)\, dp$$

which contradicts (14) since $C(q)$ is computed directly from (15) as the right hand term of formula (14). Q.E.D.

APPENDIX 2

Proof of Theorem 1

Step 1. We let the reader check that a random order value meets all four axioms listed in the Theorem; clearly the SS method meets symmetry as well. To prove the converse statement, we fix throughout the rest of the proof a *cos sharing method* x in $\mathscr{C}$: it is represented by n measures $\mu_1, ..., \mu_n$ and formulas (3) (4). We examine the consequence on these measures of SI in Step 2 and of DM in Step 3. The rest of the proof is in Step 4.

Step 2. (Characterization of SI) For every $q \in \mathbb{R}^n_+$, $\lambda > 0$ and $i \in \{1, ..., n\}$, for every measure ν on $[0, q]$, we denote by $\tau^i_\lambda[\nu]$ the image of ν on $[0, \tau^i_\lambda(q)]$ in the expansion τ^i_λ. It is defined as follows: for every continuous function F on $[0, q]$:

$$\int_{[0,\, q]} F(p)\, d\mu(p) = \int_{[0,\, \tau^i_\lambda(q)]} F(\tau^i_{1/\lambda}(p))\, d\tau^i_\lambda[\nu](p). \tag{16}$$

We claim that the cost sharing method x satisfies SI (in addition to ADD and DUM) if and only if for all q, all $\lambda > 0$ and all i s.t. $q_i > 0$ we have

$$\begin{aligned} \tau^i_\lambda[\mu_i(q)] &= \frac{1}{\lambda} \cdot \mu_i(\tau^i_\lambda(q)) \\ \tau^i_\lambda[\mu_j(q)] &= \mu_j[\tau^i_\lambda(q)] \qquad \text{for all} \quad j \neq i. \end{aligned} \tag{17}$$

To check the claim is straightforward: denote $\nu = \tau^i_\lambda[\mu_i(q)]$ and $q' = \tau^i_\lambda(q)$, and apply (19) to $F = \partial_i C$:

$$\int_{[0,\, q]} \partial_i C(p)\, d\mu_i(q)(p) = \int_{[0,\, q']} \partial_i C(\tau^i_{1/\lambda}(p))\, d\nu(p).$$

On the other hand, SI reads

$$\int_{[0,\, q]} \partial_i C(p)\, d\mu_i(q)(p) = \int_{[0,\, q']} \frac{1}{l}\, \partial_i C(\tau^i_{1/\lambda}(p))\, d\mu_i(q')(p)$$

and the equality $\nu = (1/\lambda)\, \mu_i(q')$ follows because $\partial_i C$ can be any continuously differentiable function on $[0, q]$. The second part of (17) is proven similarly.

Step 3. (Characterization of DM) We claim that the cost sharing method x satisfies DM (in addition to ADD and DUM) if and only if for all $q \in \mathbb{R}^n_+$ and all $i \in \{1, ..., n\}$ we have:

$$\forall q'_i \quad \{q'_i > q_i\} \Rightarrow \mu_i(q) \text{ and } \mu_i(q \backslash^i q'_i) \quad \text{coincide on } [0, q]. \tag{18}$$

To check "if", we write $q' = (q \backslash^i q'_i)$ and $\Delta = [0, q'] \backslash [0, q]$. If $q'_i > q_i$ and if (18) holds, we compute

$$x_i(q'; C) = \int_{[0,\, q']} \partial_i C\, d\mu_i(q') = x_i(q; C) + \int_\Delta \partial_i C\, d\mu_i(q').$$

Therefore $x_i(q; C) \leqslant x_i(q'; C)$ follows as $\partial_i C$ and $\mu_i(q')$ are nonnegative. To prove "only if", we fix q, i, and q'_i as in the premises of (18); we write

$q' = (q \backslash^i q'_i)$; we also pick an arbitrary non negative and continuously differentiable function F with domain $[0, q]$. For any ε such that $q_i < q_i + \varepsilon < q'_i$ we can extend F to a non negative and continuously differentiable function with domain $[0, q']$ and such that F is zero outside $[0, (q \backslash^i q_i + \varepsilon)]$. As noted in Step 1 of the proof of Lemma 3, there is a function C in Γ_+ such that $F = \partial_i C$. Thus we can apply demand monotonicity to q and q'. Denoting $\Delta(\varepsilon) = [0, (q \backslash^i q_i + \varepsilon)] \backslash [0, q]$ we get

$$\int_{[0,q]} F \, d\mu_i(q) \leqslant \int_{[0,q']} F \, d\mu_i(q') = \int_{[0,q]} F \, d\mu_i(q') + \int_{\Delta(\varepsilon)} F \, d\mu_i(q').$$

Letting ε go to zero

$$\int_{[0,q]} F \, d\mu_i(q) \leqslant \int_{[0,q]} F \, d\mu_i(q').$$

As F can be any non negative and continuously differentiable function, an obvious approximation argument shows that $\mu_i(q)$ is bounded above by $\mu_i(q')$ on $[0, q]$. In view of (4) these two measures have the same weight on $[0, q]$, whence they coincide as desired.

Step 4. (End of the proof.) We fix an agent i and $q \in \mathbb{R}^n_+$ such that $q_j > 0$ for all j. Pick an arbitrary Borel set Ω in $[0, q_{-i}]$ (where q_{-i} obtains from q by deleting q_i) and a number p_i, $0 < p_i \leqslant q_i$. Denote $\lambda = q_i / p_i$ and $q' = \tau^i_\lambda(q)$. Applying successively (16) (17) and (18) we get

$$\begin{aligned}\mu_i(q)([0, p_i] \times \Omega) &= \tau^i_\lambda[\mu_i(q)]([0, q_i] \times \Omega) \\ &= \frac{1}{\lambda} \mu_i(q')([0, q_i] \times \Omega) = \frac{1}{\lambda} \mu_i(q)([0, q_i] \times \Omega).\end{aligned}$$

Thus the conditional measure of $\mu_i(q)$ on $[0, q_i]$ given Ω satisfies $v([0, p_i]) = \alpha \cdot p_i$ for some constant α: it is proportional to the Lebesgue measure ℓ. We have proven that $\mu_i(q)$ is the product of ℓ on $[0, q_i]$ and a probability measure $v_i(q)$ on $[0, q_{-i}]$ (that total mass of $v_i(q)$ is 1 follows from (4)). Property (18) implies at once that $v_i(q)$ is independent of q_i: thus we write $v_i(q_{-i})$.

In order to pin down the form of the measures $v_i(q_{-i})$, we invoke budget balance. Fix a vector q and an integer m, $m \leqslant n$. Assume that q_i is positive for all $i = 1, ..., m$ and denote $T = \{m+1, ..., n\}$ and $q^* = (q \backslash^T 0)$ (note that T is empty if $m = n$). We consider the function C of the form $C(q) = f_1(q_1) \times \cdots \times f_m(q_m)$ and we compute for all $i = 1, ..., m$

$$x_i(q^*; C) = \int_{[0, q^*]} f_i'(p_i)\, f_1(p_1) \cdots f_m(p_m)\, dp_i\, dv_i(q^*_{-i})(p^*_{-i})$$

$$= f_i(q_i^*) \int_{[0, q^*_{-i}]} f_1(p_1) \cdots f_{i-1}(p_{i-1})$$

$$\times f_{i+1}(p_{i+1}) \cdots f_m(p_m)\, dv_i(q^*_{-i})(p^*_{-i}).$$

For each $j = 1, ..., n$, we fix a number r_j, $0 \leqslant r_j < q_j$. We pick ε such that $r_j + \varepsilon < q_j$ and the functions f_j such that

$$f_j = 0 \qquad \text{on } [0, r_j]; \qquad f_j = 1 \qquad \text{on} \quad [r_j + \varepsilon, q_j], \quad 0 \leqslant f_j \leqslant 1.$$

As ε goes to zero the above integral converges to $v_j(q^*_{-j})(\,]r^*_{-j}, q^*_{-j}])$ where the notation $p \in \,]a, b]$ means $a_i < p_i \leqslant b_i$ for all i. In the limit, budget balance yields

$$\sum_{j=1}^{m} v_j(q^*_{-j})(\,]r^*_{-j}, q^*_{-j}]) = 1.$$

As each $v_j(q^*_{-j})$ is a positive measure, the weight of $]r_j^*, q^*_{-j}]$ must be independent of r_{-j}. Therefore denoting

$$\Omega_i(q; T) = \{ p_{-i} \mid 0 < p_j \leqslant q_j \text{ for all } j \neq i, j \notin T;\ p_j = 0$$
$$\text{for all } j \in T;\ p_{-i} \neq q^*_{-i} \}.$$

we conclude that $v_i(q_{-i})(\Omega_i(q; T)) = 0$. Clearly our choice of $T = \{m+1, ..., n\}$ did not bring any loss of generality so the conclusion holds for any (possibly empty) proper subset T of $\{1, ..., n\}$ and any $i \notin T$ (note that if $q_j = 0$ for some $j \notin T$, the set $\Omega_i(q; T)$ is empty).

Varying T over all proper subsets of $\{1, ..., n\}$ it follows that $v_i(q_{-i})$ can only charge the stand alone points $(q_{-i} \backslash^T 0)$. By scale invariance (17) the weight of any such point must be independent of q_{-i} (provided all the coordinates q_j remain strictly positive). Thus the probability measure $v_i(q_{-i})$ takes the form

$$v_i(q_{-i}) = \sum_{\varnothing \neq S \subseteq N \backslash i} \alpha_i(S) \cdot \delta_{(q_{-i} \backslash^{S^c} 0)} \qquad \text{where} \quad S^c = \{1, ..., n\} \backslash (S \cup \{i\}). \tag{19}$$

As the measure $\mu_i(q)$ is the product of $v_i(q_{-i})$ with the Lebesgue measure on $[0, q_i]$, the above formula combined with (3) gives

$$x_i(q; C) = \sum_{\varnothing \neq S \subseteq N \backslash i} \alpha_i(S) \cdot (c(S \cup \{i\}; q) - c(S; q)).$$

This formula holds for all q such that $q_j > 0$ for all j. By continuity at zero, it holds for all $q \in \mathbb{R}^n_+$ as well. Given budget balance, this is precisely the general form of the random order value, as shown by [56, Theorems 12 and 13]. The Corollary follows at once from Weber's Theorem 15. Q.E.D.

Remark 5. If we drop the requirement of continuity at zero, the above proof is easily adapted to characterize the methods in $\mathscr{C}$ meeting SI and DM. They coincide with a random order value on every subset of $\mathbb{R}^n_+$ where the set of nonzero coordinates remains constant, but these random order values are not consistent with one another. That is to say, for each nonempty subset T of N, we can choose an arbitrary random order value x^T among the agents of T, and this defines the cost sharing method x over the subset $P(T)$ defined by $q_i > 0$ if and only if $i \in T$ (of course, agents outside T have $q_j = 0$ and $x_j = 0$).

APPENDIX 3

Proof of Theorem 2

The proof that serial cost sharing meets the upper bound axiom is given immediately after the statement of the theorem. That it meets demand monotonicity is equally straightforward hence omitted.

We prove now the implication $ii \Rightarrow i$ in three steps. First we show, using demand monotonicity that for any i and p such that $p_j > p_i$ for some $j \neq i$, that $\mu_i(q)$ is 0 for any small neighborhood of p. Next we show, combining the previous result with budget balance, that for any p such that $p_i > p_j$ for some $j \neq i$, that $\mu_i(q)$ is similarly "nil" at p. Finally, combining these two steps shows that $\mu_i(q)$ is "nil" at any point where the measure for the serial mechanism is "nil." Thus the supports of the two must coincide. Budget balance then requires that the two methods have the same measure and must therefore lead to identical mechanisms.

The following approximation of the unit step function will be useful below: let $\Theta_\varepsilon(x) = \max[0, \min(x/\varepsilon, 1)]$, where Θ_ε is 0 for $x \leqslant 0$, 1 for $x \geqslant \varepsilon$, and linear in between.

Step 1. Fix i and q. Now consider any $p \in [0, q]$ such that $p_i < p_j$ for some $i \neq j$. Define $q'_i = p_i$ and choose $\varepsilon > 0$ sufficiently small such that $p_i + \varepsilon < p_j$. Set $t = p_i + p_j - \varepsilon/2$.

Consider the homogeneous cost function $C(q) = \Theta_\varepsilon(q_i + q_j - t)$, which is approximately a homogeneous step function: $C(q) = 0$ if $q_i + q_j \leqslant t$,

$C(q)=1$ if $q_i+q_j \geqslant t+\varepsilon$, and the function is linear in between with slope $1/\varepsilon$.[15]

By assumption $x_i(q_i', q_{-i}) \leqslant C(q_i'e) = 0$ since $2q_i' < t$, and thus $\int_{[0,(q_i';q_{-i})]} \partial_i C(p)\, d\mu_i(q_i';q_{-i})(p)=0$. Since $\partial_i C(\cdot)=1/\varepsilon$ on $A\times[0,q_{-ij}]$ where A is any sufficiently small neighborhood of (p_i, p_j) this implies that $\mu_i(q_i', q_{-i})(A\times[0,q_{-ij}])=0$. By demand monotonicity (see (18)), this implies that $\mu_i(q)(A\times[0,q_{-ij}])=0$.

Step 2. Now for any i, q consider p such that $p_i > p_j$ for some j and $p_j < q_j$. Let $C(z)=D(z)+F(z)$ where $D(z)=(\Theta_\varepsilon(z_j-p_j+\varepsilon/2)-\Theta_\varepsilon(z_j-p_j-\varepsilon/2))\,\Theta_\varepsilon(z_i-p_i+\varepsilon/2)$ and $F(z)=\Theta_\varepsilon(z_j-p_j-\varepsilon/2)$, which is constructed so that $\partial_i D(z)=1/\varepsilon$ in any sufficiently small neighborhood of (p_i, p_j) and $\partial_j D(z)=0$ when $z_i \leqslant z_j$. Note that $F(z)$ only depends on z_j and thus player j pays all of $F(z)$ by additivity and dummy ($F(z)$ is only included to make $C(z)$ nondecreasing). Consider $x_j(q; C)$. By the argument in Step 1 we must have that $x_j(q;C)=F(q)$ since $\mu_j(q)$ is nil on the support of $\partial_j D(\cdot)$. Thus by budget balance $x_i(q;C)=0$ since $D(q)=0$, but we know that $x_i(q;C)=\text{int\_}[0,q]\ \partial_i D(z)\, d\mu_i(q)(z) \geqslant \mu_i(q)(A\times[0,q_{-ij}])/\varepsilon$. where A is a sufficiently small neighborhood of (p_i, p_j). This implies that $\mu_i(q)(A\times[0,q_{-ij}])=0$.

Step 3. Completion of Proof. Thus $\mu_i(q)(A\times[0;q_{-ij}])=0$ where A is any small enough neighborhood of p where for some $j\neq i$, $p_i\neq p_j$ and $p_j<q_j$ if $p_i>p_j$. Then $\mu_i(q)(B)=0$ for any neighborhood B included in $B\subset A\times[0,q_{-ij}]$. This shows that the support of $\mu_i(q)$ must lie along the serial "path," and a computation (similar to the ones above) show that budget balance ensures that this is precisely the same measure that defines the serial method, and thus the two methods must be the same.

APPENDIX 4

Proof of Lemma 4

Proof of Statement (i). Let x be a cost sharing method in $\mathscr{C}$ equal to average cost pricing for homogeneous cost functions, and demand monotonic. We derive a contradiction by means of a counterexample in the case $n=2$ (this is clearly enough).

[15] Note that $C(\cdot)$ is continuous and differentiable almost everywhere; that it is not continuously differentiable and therefore it is not technically an acceptable cost function. However, it can be "smoothed" to a continuously differentiable function which is close in the norm of uniform convergence and works equally well in the remainder of the proof. We omit the straightforward details.

Consider the function C in Γ^h_+:

$$C(q) = C_0(q_1 + q_2) \quad \text{where} \quad C_0(z) = z^2 \quad \text{if } z \leqslant 1, = 2z - 1 \quad \text{if } 1 \leqslant z.$$

Denote $q = (1, 1)$ and $q' = (2, 1)$. By assumption $x_1(q; C) = 1.5$; $x_1(q'; C) = 3.33$. On the other hand, the measures $\mu_1(q)$ (in Lemma 3) satisfy (18) because x is demand monotonic; thus,

$$x_1(q'; C) - x_1(q; C) = \int_\Delta \partial_1 C \, d\mu_1(q') = 2\mu_1(q')(\Delta),$$

where Δ is the rectangle $[0, q'] \backslash [0, q]$. By property (4) $\mu_1(q')(\Delta) = 1$, which is the desired contradiction. Q.E.D.

Proof of Statement (ii). For the cost function $C(q) = (q_N)^2$, an element of Γ^h_+, the properties ACP and UBH are clearly incompatible.

Proof of Statement (iii). Again we derive the contradiction by an example for $n = 2$. Let $C(q_1, q_2) = (q_1 + q_2)^2/2$. Notice that $x_1(q; C) = q_1^2/2 + x_1(q; \hat{C})$ where $\hat{C}(q) = q_1 q_2$ and $\partial_1 \hat{C}(q) = q_2$. Since either $x_1(e; \hat{C}) > 0$ or $x_2(e; \hat{C}) > 0$ we will assume w.l.o.g. that $x_1(e; \hat{C}) = \gamma > 0$. Now assume that $q_1 = 1$. By the representation lemma,

$$x_i(q; \hat{C}) = \int_{[0,\, q]} p_2 \, d\mu_1(q)(p),$$

and by scale invariance (using the characterization in the proof of Theorem 1) we get

$$x_1(q; \hat{C}) = q_2 \int_{[0,\, e]} p_2 \, d\mu_1(e)(p)$$

but this is simply $q_2 x_1(e; \hat{C})$ and therefore $x_1(q; C) = (q_1^2/2) + q_2 \gamma$ which is unbounded in q_2.

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