Strong Monotonicity in Surplus Sharing

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Abstract

We consider three new axioms for surplus sharing problems. The first is strong monotonicity which says that workers should be rewarded for increases in productivity and the second says that productive workers should receive some compensation. The third requires that the surplus sharing rule should be well defined (and continuous on) the set of threshold functions. We show that none of the standard "equitable" mechanisms satisfy either of these axioms and then present a constructive characterization of mechanisms which do. Using this we construct several new mechanisms. These are the Almost Flat mechanism, the Spread Aumann-Shapley mechanism, and the Spread Serial mechanism, which have many desirable properties.

1 Introduction

The sharing of joint costs or surplus arise in a large number of situations. Surplus sharing problems often arise from the sharing of profits in joint ventures [14, 18, 25, 26]. Formally equivalent are cost sharing problems in which the cost of serving agents' demands must be divided. These problems range from allocating telephone costs in a university [4] to sharing the costs for public goods [17, 7], such as electricity [16], to allocating cost [12, 13] or congestion on the Internet [8, 28, 29], to sharing the cost for research consortia [1].

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Throughout this paper we will stick to the surplus sharing interpretation as our axioms are somewhat more compelling in that context, although our results are applicable in the cost sharing setting.

In this paper we consider several new axioms in the study of surplus sharing mechanisms. These axioms are: strong monotonicity, converse dummy, and continuous extendibility to the threshold functions. We show that many of the important surplus sharing mechanisms do not satisfy these axioms and then construct several new mechanism which do, based on a constructive characterization of these axioms.

We now give the motivation behind these axioms, which are most easily described in the cooperative production context. Consider a cooperative enterprise in which there are N participants, and each participant determines the amount of "effort" which they put into the enterprise. The total profit is a (nondecreasing) function of this vector of efforts.[1] A surplus sharing mechanism is a rule for dividing up the profits, as a function of efforts and the production function. In this setting, strong monotonicity is the requirement that if a participant's productive ability strictly increases (perhaps through additional training) then that participant should receive more. One can also interpret this as requiring that gains in technology are distributed fairly. This is a natural extension of Young's monotonicity axiom for cooperative games [32, 33].

Our second axiom, converse dummy is implied by strong monotonicity. It requires that if a participant puts in positive valuable effort, then that person must receive at least some

[1] Note that effort could be the number of hours of labor or even a physical contribution, such as an input to production.

of the profits. As we will see, in the additive setting this axiom is equivalent to the first one.

Our third axiom, continuous extendibility to the threshold functions, requires that the surplus sharing mechanism be well defined on an important class of production functions, the threshold functions. These are functions of the form $T_\alpha(x) = 1$ if $x_i \geq \alpha_i$ for all i and 0 otherwise. In the production context these are quite common as many goods are binary, they either exist or they don't.

This paper is organized as follows: in Section 2 we provide a brief introduction to surplus sharing mechanisms, including a review of common mechanisms and standard axioms. Then in Section 3 we introduce our new axioms and discuss their relationships with standard mechanisms. In Section 4 we provide constructive characterizations of these and which we use to construct three new methods which satisfy them and also have other desirable properties. We then discuss, in Section 5, the difficulties involved in axiomatizing these new methods, and conclude in Section 6 with a brief summary of our results.

2 Surplus Sharing: Definitions

We consider a surplus sharing problem with a fixed number of participants, $i \in N = \{1, 2, \ldots, n\}$. Each player contributes a level of effort to the project, $e_i \in \Re_+$. Given the vector of inputs, e, the project has a certain value $V(e)$ which is then divided among the participants, where $V(\cdot) \in \mathcal{V}$, the set of all nondecreasing (once) continuously differentiable functions satisfying $V(0) = 0$. The division rule is denoted by $x \in \mathcal{SS}$, where $x_i(e; V)$ is the share of the value allocated to participant i. We assume that participants cannot have negative shares, $x_i(e; V) \geq 0$, and that the total amount allocated must exactly equal the profit, $\sum_{i=1}^{n} x_i(e;V) = V(e)$, which is often denoted "efficiency".

2.1 Standard Axioms

In addition to the basic constructs, there is a large body of literature on additional properties that sharing methods 'should' satisfy. (See [23] for a survey and an extensive bibliography.) In this section we provide a brief survey of some of the common/important ones which will be relevant to our analysis.

The first axiom was first used by Shapley [27] to characterize the Shapley value. In the surplus sharing interpretation, it says that a participant who's effort adds no value to the project should receive none of the value of the project.

Definition 1 (Dummy)

1) Player i is a "dummy participant," if for all $p \in \Re^n_+$, $\partial_i V(p) = 0$.

2) A surplus sharing mechanism satisfies the "Dummy axiom" (DUM) if whenever player i is a dummy, $x_i(e;V) = 0$.

Our second axiom is nearly universal in the surplus (and cost) sharing literature. (With the notable exception of [31].) It is usually motivated by arguments of decentralization.

Definition 2 (Additivity) *A surplus sharing mechanism satisfies "Additivity" (ADD) if $x_i(e;V_1+V_2) = x_i(e;V_1) + x_i(e;V_1)$, for all $i \in N$, $e \in \Re^n_+$ and $V_1, V_2 \in \mathcal{V}$.*

Note that for cooperative games, additivity and dummy along with other basic axioms uniquely define the Shapley value; however, in the continuous surplus sharing problem there is a large class of methods which satisfy them [10].

Our next axiom is well known. Scale invariance (SI), is the statement that the relative scales used in defining the e_i's are irrelevant to the shares, while the following axiom, demand monotonicity (DM) [22, 10], is a basic incentive constraint which requires that participants not be penalized for increasing their level of input.

Definition 3 (Scale Invariance) *Given $\lambda \in \Re^n_{++}$, define $\tau_\lambda(e)$ by $\tau_\lambda(e)_i = \lambda_i e_i$ for and define $\tau_\lambda(V)$ by $\tau_\lambda(V)(e) = V(\tau_\lambda(e))$, for $V \in \mathcal{V}$. A SSM, $x \in \mathcal{SS}$, is scale invariant if $x(\tau_\lambda(e); V) = x(e; \tau_\lambda(V))$, for all $\lambda \in \Re^n_{++}$ and $V \in \mathcal{V}$.*

Definition 4 (Demand Monotonicity) *A SSM is demand monotonic if for all $e, e' \geq 0$ such that $e_i \leq e'_i$ and $e_{-i} = e'_{-i}$ and all $V \in \mathcal{V}$, $x_i(e; V) \leq x_i(e'; V)$.*

2.2 Examples: Surplus Sharing Mechanisms

In this section we review a number of well known SSM's.

Equal Share (ES): Perhaps the simplest cost sharing mechanism is the "Equal Share" mechanism [21], $x_i^{ES}(e; V) = V(e)/n$. However, this mechanism has many undesirable properties; in particular, it ignores the levels of input, so that a participant who supplies a large amount of input receives exactly the same share as one who supplies no input.

Proportional Share (PS): A simple generalization of this method is the "Proportional Share" method [21], $x_i^{PS}(e; V) = e_i V(e)/|e|$, where $|e| = \sum_{i=1}^n e_i$, which allocates shares in proportion with participants level of input. However, this mechanism still does not take into account the 'value' of a participant's input. For example, suppose that participant 1's input is valueless, e.g., $V(e_1, e_2) = 100e_2$, then if participant 1 chooses $e_1 = 99$ and participant 2 chooses $e_2 = 1$ then participant 1 gets 99 and participant 2 gets 1 even though participant 1 contributed nothing of value.

Essentially both ES and PS violate the dummy axiom and the combination of dummy and additivity restrict the set of mechanisms to those which are "equitable" according to these and related arguments.

Shapley-Shubik (SS): The "Shapley-Shubik" method was introduced in [30]. It arises from the application of the Shapley Value [27] to the derived cooperative game. Before we present its formula, we first describe the "Incremental" mechanism for order ψ. Let ψ be an ordering of the participants, i.e., $\psi \in \Psi$ is a bijection from the set of players to the set $\{1, 2, ..., n\}$. Then the formula for this method is

$$x_i^{\psi}(e, V) = V(e_{S_{\psi}^{+}(i)}, 0_{-S_{\psi}^{+}(i)}) - V(e_{S_{\psi}^{-}(i)}, 0_{-S_{\psi}^{-}(i)}),$$

where $S_{\psi}^{+}(i) = \{j \mid \psi(j) \leq \psi(i)\}$ and $S_{\psi}^{-}(i) = \{j \mid \psi(j) < \psi(i)\}$. The Shapley-Shubik method is the average over all incremental methods

$$x_i^{SS}(e; C) = (n!)^{-1} \sum_{\psi \in \Psi} x_i^{\psi}(e; C).$$

This method satisfies additivity, dummy, scale invariance and demand monotonicity and is fact characterized by a combination of these four axioms [10].

Aumann-Shapley (AS): The Aumann-Shapley mechanism was proposed as a cost sharing mechanism in [6] and then axiomatized by economic axioms in [5, 20]. It is given by the 'diagonal' formula

$$x_i^{AS}(e, V) = \int_0^1 \partial_i V(te)(t^{w_i - 1} w_i e_i) dt$$

which arises by applying the Shapley value to surplus sharing problem when viewed as a nonatomic game [2]. We will also be interested in weighted versions of it which were analyzed in [19]. These are defined by a weight vector $w \in \Re^N_{++}$ and given by the formula

$$x_i^{AS(w)}(e, V) = \int_0^1 \partial_i V(h^w(t, e))e_i dt,$$

where $h^w(t, e)_i = t^{w_i} e_i$. Any weighted version satisfies dummy, additivity, and scale invariance, but not demand monotonicity.

Serial (SER): The Serial method was introduced in [29, 24] for the case of homogeneous functions $V(e) = \hat{V}(|e_i|)$ and extended to nonhomogeneous functions in [10]. It is given by

$$x_i^{SER}(e, V) = \int_0^{e_i} \partial_i V(t\mathbf{1} \wedge e)dt,$$

where $\mathbf{1}$ is the unit vector and $x \wedge y$ is greatest lower bound of x and y, e.g., $(1, 4) \wedge (3, 2) = (1, 2)$. Later in the paper we will be interested in a weighted version which we define as follows. Given a weight vector $w \in \Re^N_{++}$ define the weighted serial method by

$$x_i^{SER(w)}(e, V) = \int_0^{e_i} \partial_i V(g^w(t, e))w_i dt,$$

where $g^w(t, e)_i = (tw_i) \wedge e_i$. Any weighted version satisfies dummy, additivity, and demand monotonicity, but not scale invariance.

2.3 Path Methods

One insight that will be useful later, is that many of these methods can be written as an integral over a path. For example, the Aumann-Shapley method can be written as

$$x_i^{AS}(e, V) = \int_0^{\infty} \partial_i V(\gamma(t; e))d\gamma_i(t; e)$$

where $\gamma(t;e) = \min[t,1]e$, while the serial method is given by an analogous formula with $\gamma(t;e) = (t\mathbf{1}) \wedge e$. Also, any incremental method is also generated by a path.

Definition 5 (Path) *A path function γ is a mapping $\gamma : [0,\infty] \times \Re^n \to \Re^n$ satisfying the following for each $e \geq 0$:*

1) $\gamma(t;e)$ is continuous and nondecreasing in t.

2) $\gamma(0;e) = 0$ and there exists a $\hat{t} > 0$ such that for all $t \geq \hat{t}$, $\gamma^{AS}(t;e) = e$.

Let the set of all such path functions be denoted Γ.

Given any path we can define the related surplus sharing mechanism.

Path Methods: Given a path function γ the path mechanism generated by γ is defined by:

$$x_i^\gamma(e,V) = \int_0^\infty \partial_i V(\gamma(t;e))d\gamma_i(t;e)$$

On interesting result is that the path methods "generate" all the additive methods:[2]

Proposition 1 ([9]) *The following are equivalent:*

i) $x \in \mathcal{SS}$ satisfies dummy and additivity.

ii) There exists a family of probability measures, indexed by e, μ^e each on $\Gamma(e)$, where $\Gamma(e)$ is the restriction of Γ to a specific e, such that

$$x = \int_{\gamma \in \Gamma(e)} x^\gamma d\mu^e(\gamma).$$

Although we will not use this result directly, we will construct a variety of methods using the formula in part (ii) of the proposition.

[2] An analogous result for nonatomic games was proven in [11].

3 Strong Monotonicity, Converse Dummy and Threshold Functions

In this section we present three new properties that seem desirable in a surplus sharing mechanism.

3.1 Strong Monotonicity and Converse Dummy

Consider a worker who gets additional training, such that she is more efficient at her job. One would naturally expect that her pay would not decrease. Similarly, a firm which improves the quality of their input should also not expect to receive less from the joint project. This is the idea behind monotonicity. (This concept was introduced by Young [32] for the case where e_i is binary.) However, if the increased productivity actually increases the total profit of the project, then one might expect the pay to actually increase. This is the new axiom of strong monotonicity.

Definition 6 (Productivity) *Given two functions V and V', participant i is said to be more productive under V than V' at e if $\partial_i V(p) \geq \partial_i V'(p)$ for all $p \in [0, e]$ and strictly more productive if, in addition, $\exists p' \in [0, e]$ such that $V(p') > V'(p')$.*

Definition 7 (Monotonicity and Strong Monotonocity) *A SSM is* monotonic *if whenever participant i is more productive under V than V' at e, $x_i(e; V) \geq x_i(e; V')$. It is* strongly monotonic *if in addition whenever participant i is strictly more productive under V than V' at e, $x_i(e; V) > x_i(e; V')$.*

Theorem 1

1) The following methods are monotonic: Equal and Proportional Share, Shapley-Shubik, Aumann-Shapley and Serial.

2) All methods satisfying additivity and dummy are monotonic.

3) The Equal and Proportional Share methods are strongly monotonic while the following are not: Shapley-Shubik, Aumann-Shapley and Serial.

Proof: Part (2) was proven in [10]. This immediately proves statement (1) for the additive methods: Shapley-Shubik, Aumann-Shapley and Serial. Lemma 2 (below) proves statement (3) for the additive methods. Statement (1) for Equal and Proportional Share follow directly from their definitions. □.

Of all the methods considered only two are strongly monotonic; however, these are typically considered the least palatable methods. Thus our goal will be to construct strongly monotonic methods which satisfy dummy and other interesting axioms.

Closely related to strong monotonicity is the "converse dummy" axiom,[3]

Definition 8 *A SSM x satisfies the* converse dummy *axiom, if for any e with $e_i > 0$ and V such that player i is not a dummy player and $\partial_i V(p) > 0$ for some $p \in [0, e]$, $x_i(e; V) > 0$.*

Thus, converse dummy implies that any agent who contributes positive effort and by doing so increases the profit should receive some compensation. Note that strong monotonicity

[3]This axiom was suggested to me by Y. Sprumont.

implies converse dummy in general. Interestingly, under dummy and additivity, the reverse implication holds.

Theorem 2 *Let x satisfy additivity and dummy. Then x satisfies converse dummy if and only if it is strongly monotonic.*

Proof: Assume that x satisfies dummy, additivity and converse dummy. Consider some i, e, V, V' where i is strictly more productive under V than under V' at e. Then $x_i(e, V) - x_i(e, V') = x_i(e, V - V' + W) - x_i(e, W)$ by additivity, where $W(e) = k|e|$ and k is chosen to be sufficiently large so that $V - V' + W \in \mathcal{V}$. Note that dummy and additivity combine to imply that $x_i(e, W) = k_i$. Now, as proven in [10], for any $V \in \mathcal{V}$, $x_i(e, V) = T(\partial_i(V - V' + W))$ where T is a positive linear functional (on the linear vector space generated by $\mathcal{V}$) and thus $x_i(e, V) = T(\partial_i(V - V')) + T(\partial_i W)$. Now consider the function $Y(p) = \int_0^{p_i} \partial_i(V - V')(p_{-i}, s)ds$. By converse dummy $x_i(e; Y) > 0$, but since $x_i(e; Y) = T(\partial_i(V - V'))$ this implies that $T(\partial_i(V - V')) > 0$ which implies that $x_i(e, V) = x_i(e; Y) + k_i$. Combining these prove that x is strongly monotonic. The other direction is proven similarly. □

This leads to our next result:

Theorem 3 *The Equal and Proportional Share methods satisfy converse dummy while the following do not: Shapley-Shubik, Aumann-Shapley and Serial.*

Proof: This theorem follows immediately from the previous one. □

3.2 Threshold Functions

Most of the methods discussed so far are only formally defined for the case when the profit functions are continuously differentiable. However, this leaves out an important class of profit functions, the threshold functions. Threshold functions arise when there is a minimum effort vector needed to complete a specific task. For example, in the production context, typically there is a certain set of parts needed to construct a machine, and without even one of these parts the machine will not function. Another interesting example arises in distributed computations (see, e.g., [15, 3]); in this case an incomplete computation is typically worthless. Thus, in this section we consider the extension of SSMs defined on continuously differentiable functions to the set of threshold functions. We will require that this extension be continuous.

For any $\alpha \geq 0$ let T_α represent a threshold function: $T_\alpha(e)$ is 1 if $e \geq \alpha$ and 0 otherwise. Let $\mathcal{T}$ be the set of all threshold functions and define $\hat{V} = \{V \in \mathcal{V} \oplus \mathcal{T} \mid V \text{ is monotone}\}$. An extension $\hat{x} \in \mathcal{SS}(\mathcal{T})$ of $x \in \mathcal{SS}$ is a SSM defined for all $V \in \hat{V}$.

Definition 9 *A mechanism $\hat{x} \in \mathcal{SS}(\mathcal{T})$ is a continuous threshold extension (CTE) of $x \in \mathcal{SS}$ if the following hold:*

i) $x(e;V) = \hat{x}(e;V)$ for all $V \in \mathcal{V}$

ii) For any sequence $V^1, V^2, \ldots$ with $V^k \in \hat{V}$ which satisfies $\lim_{k\to\infty} V^k(p) = T_\alpha(p)$ for all $p \neq \alpha$ (pointwise convergence), $\lim_{j\to\infty} \hat{x}(e;V^j) = \hat{x}(e;V)$, for $e \neq \alpha$.

Note that a CTE is smooth in the sense that small changes in α do not cause large changes in the allocation, when these changes do not affect the total payoff to be allocated.

Lemma 1 *Any CTE, $\hat{x} \in \mathcal{SS}(\mathcal{T})$ of some $x \in \mathcal{SS}$ is α-continuous, i.e., $x(e; T_\alpha)$ is continous in α when $e \neq \alpha$ for all $\alpha \in \Re^\infty_+$.*

Proof: Obvious from statement (ii) in the definition of a CTE. $\diamond$

Using this, we now show that none of the standard methods satisfying dummy have CTE's, as was the case for strict monotonicity.

Theorem 4 *The following SSM's do not have any CTE's: Aumann-Shapley, Shapley-Shubik, and Serial, while Equal share and Proportional share do.*

Proof: The formal proof for the first 3 methods requires the use of Lemma 3 below. However, we provide the intuition here, showing that the obvious extensions violate α-continuity. Consider projects with two participants; For the Shapley-Shubik method $x_1^{SS}(\mathbf{1}; T_{0,1/2}) = 0$ while $x_1^{SS}(\mathbf{1}; T_{\epsilon,1/2}) = 1/2$ for any $\epsilon \in (0, 1)$. For the Aumann-Shapley method consider $x_1^{AS}(\mathbf{1}; T_{1/2+\epsilon,1/2-\epsilon})$ which is 0 for $\epsilon \in (0, 1/2)$ and 1 for $\epsilon \in (-1/2, 0)$; these shares are identical for the Serial method. Lastly, note that neither Equal share nor Proportional share depend on α when $\alpha < e$. $\square$

4 Strongly Monotonic Methods with CTEs

In this section we will construct three new methods that satisfy strong monotonicity, two of which also have unique CTE's, while also satisfying additivity and dummy. These methods are based on the intuition from Proposition 1 and are somewhat more complex than the well known additive methods. The idea behind their construction is given by the following results.

4.1 Characterizations under Additivity and Dummy

Definition 10 *Given i, $p \in [0, e]$ and an open neighborhood η of p let $G(i, p, \eta)$ be the set of paths which pass through η with nonzero slope for player i, i.e., for each $\gamma \in G(i, p, \eta)$ there exist t, t' such that $\gamma_i(t; e) > \gamma_i(t'; e)$ and both $\gamma_i(t; e)$ and $\gamma_i(t'; e)$ are in η.*

Now we can state our main lemmas.

Lemma 2 *Let x be constructed as a sum over paths, $x(e; V) = \int_{\gamma \in \Gamma(e)} x^\gamma d\nu^e(\gamma)$. Then, $x(e; \cdot)$ is strongly monotonic if and only if for all $e \in \Re^n_+$, $i \in N$, $p \in [0, e]$ and open neighborhood of p, η, $\nu^e(G(i, p, \eta); e) > 0$.*

Proof: $\Leftarrow$: Suppose that participant i is more productive under V than V' at $e > 0$. Thus, there exists some $p \in [0, e)$ such that $\partial_i V(p) > \partial_i V'(p)$ and by continuity there exists an open neighborhood η of p such that for all $p' \in \eta$, $\partial_i V(p') \geq \partial_i V'(p') + \epsilon$ for some $\epsilon > 0$. Now for each $\gamma \in G(i, p, \eta)$, define $\lambda(\gamma) = \max_{t,t'} |\gamma_i(t; e) - \gamma_i(t'; e)|$ and note that $x^\gamma(e; V) - x^\gamma(e; V') > \epsilon\lambda(\gamma)$. Since, for all $\gamma \in G(i, p, \eta)$, $x^\gamma(e; V) - x^\gamma(e; V') > \epsilon\lambda(\gamma)$ and $\nu(G(i, p, \eta); e) > 0$, this implies that $x_i(e; V) > x_i(e; V')$ proving the lemma.

$\Rightarrow$: Consider the contrapositive. Assume that there exists some point $p \in [0, e)$ such that for some closed neighborhood η of p, $\nu(G(i, p, \eta); e) = 0$, for some SSM, x, which is represented by the measure ν on $\Gamma(e)$. Now consider the case when $V' \equiv 0$ while $V(p') = \int_0^{p'_i} B(p'_{-i}, s) ds$, where $B(p')$ is a continuously differentiable "bump function" inside η, i.e., $B(p) > 0$ on some open subset on the interior of η and 0 everywhere else. By construction participant i is more productive under V than V' at e; however, $x_i(e; V) = 0 = x_i(e; V')$ since for all paths, γ, in the support of ν, $x_i^\gamma(e;V) = 0$. $\diamond$

Lemma 3 *Let x be constructed as a sum over paths, $x(e;V) = \int_{\gamma \in \Gamma(e)} x^\gamma d\nu(\gamma)$. Then, x has a CTE if and only if for all $e \in \Re^n_+$ and $S \subseteq N$ such that $|S| \geq 2$ and $\beta \in [0,e]$ the set of paths which intersect $L_S^e(\beta) = \{p = [0,e] \mid \forall i \in S,\ p_i = \beta_i\}$ has measure 0, with respect to ν.*

Proof: $\Leftarrow$: We can construct a CTE in the following manner. For any $i \in N$ e and T_α, with $\alpha < e$ define $\hat{x}_i(e;T_\alpha) = \nu(S_\alpha)$ where $S_i^\alpha(e)$ is the set of paths for which there exists a $t > 0$ such that $\gamma_i(t;e) < \alpha_i$ and $\gamma_j(t;e) > \alpha_j$ for all $j \neq i$. By assumption, since $\nu(\sum_i S_i^\alpha(e)) = 1$, these shares add up to 1 for all α and therefore this is an element of $\mathcal{SS}(\mathcal{T})$. Now extend $\hat{x}$ to an element of $\mathcal{SS}(\mathcal{T})$ by linearity. Also, note that by the construction, $\hat{x}$ is α-continuous, since a small change in α can only change the set of intersected paths by a small amount.

Next, note that $T_\alpha(p) = \Pi_{i \in N}\Theta(p_i - \alpha_i)$ where $\Theta(s)$ is the step function which is 0 for $s < 0$ and 1 otherwise. Now consider a simple smoothing of T_α: $T_\alpha^\epsilon = \Pi_{i \in N}\Theta^\epsilon(p_i - \alpha_i)$, where $\Theta^\epsilon(s)$ is a monotone and continuously differentiable function which is 0 for $s < -\epsilon$ and 1 for $s > \epsilon$ and note that $\lim_{\epsilon \to 0} T_\alpha^\epsilon(s) = T_\alpha(s)$ for all $s \neq e$. It is straightforward to show that

$$\lim_{\epsilon \to 0} x(e; T_\alpha^\epsilon) = \hat{x}(e; T_\alpha)$$

from the construction of $\hat{x}$.

Note that the threshold functions form a basis for the infinitely continuously differentiable monotone functions $\mathcal{V}^\infty$ through the following identity:

$$V(e) = \sum_{S \subseteq N} (-1)^{|S|+1} \int_{\alpha_S \in [0,e_S]} \partial_S V(\alpha_S) T_{\alpha_S} d\alpha_S.$$

Thus, for any α, consider a sequence $\{V^k\}_{k=1}^{\infty}$, where $V^k \in \mathcal{V}^\infty$ which converges to T_α, pointwise. From this identity,

$$V^k(p) = \sum_{S \subseteq N} \int_{\alpha_S \in [0,q_S]} f^k(\alpha'_S) T_{\alpha'}(p) d\alpha'_S,$$

where for all $\alpha' \neq \alpha$, $\lim_{k\to\infty} f^k(\alpha') = 0$. Since $\hat{x}$ is linear and α-continuous

$$\hat{x}(e; V^k) = \sum_{S \subseteq N} \int_{\alpha_S \in [0,e_S]} f^k(\alpha'_S) \hat{x}(e; T_{\alpha'}) d\alpha'_S.$$

Taking the limit as $k \to \infty$ shows that $\lim_{k\to\infty} \hat{x}(e; V^k) = \hat{x}(e; T_\alpha)$. Since $\hat{x}$ is linear, and $\mathcal{V}^\infty$ is dense in $\hat{V}$ this completes the first half of the proof.

$\Rightarrow$: First consider the case when $n = 2$. In this case $S = \{1, 2\}$ and thus $L_S(\beta) = \beta$. Assume that there is some point β such that the set of paths which intersect the point β has measure $\phi > 0$. Now consider the values of $x(e; T^{\epsilon^2}_{\alpha_1-\epsilon,\alpha_2})$ and $x(e; T^{\epsilon^2}_{\alpha_1-\epsilon,\alpha_2})$. By a similar analysis as in the proof of the previous lemma, $\lim_{\epsilon\to 0^+}[x(e; T^{\epsilon^2}_{\alpha_1-\epsilon,\alpha_2}) - x(e; T^{\epsilon^2}_{\alpha_1-\epsilon,\alpha_2})] = \phi$, and thus no extension can satisfy α-continuity.

For $n > 2$ assume that for some S, β, a positive set of paths intersect $L_S(\beta)$. Choose $i, j \in S$ such that $i \neq j$. Consider T_α where $\alpha = (\beta_i, \beta_j, 0_{N\setminus\{i,j\}})$, where all participants except for i and j are dummy participants and thus receive no share. Essentially we have reduced this to a two participant problem, $\hat{x}$ with cost function T_{β_i,β_j}, where x is generated by the projected paths, where the projection of path $\gamma(t; e)$ is simply the two components $(\gamma_i(t; e), \gamma_j(t; e))$. By assumption, the subset of paths generating $\hat{x}$ which pass through the point (β_i, β_j) has a positive measure in ν. Thus, the argument for $n = 2$ applies, and no extension can be α-continuous. $\diamond$

Thus, if we take a convex combination of paths which "cover" the region $[0, e]$ then we can construct a method which is strongly monotonic; similarly, if choose a measure that doesn't create any "atoms" in $[0, e]$ we will have constructed a method that has a CTE. In the following subsection, we do this for three particularly nice sets of paths.

4.2 New SSMs

The first method we denote the "Almost Flat" (AF) method as it is the method that comes closest to covering the region $[0, e]$ evenly. The Almost Flat method is constructed as follows: Pick an ordering ψ and a point $\hat{p} \in [0, e]$. Now define the path $\gamma^{\psi,\hat{p}}$ as follows: $\gamma_i^{\psi,\hat{p}}(t; e) =:$

$$\begin{array}{rll} 0 & \text{for} & t < \psi(i) - 1 \\ \hat{p}_i(t - \psi(i) + 1) & \text{for} & t \in [\psi(i) - 1, \psi(i)] \\ \hat{p}_i & \text{for} & t \in (\psi(i), n - \psi(i) - 1) \\ \hat{p}_i + (e_i - \hat{p}_i)(t - n + \psi(i) + 1) & \text{for} & t \in [n - \psi(i)) - 1, n - \psi(i)] \\ e_i & \text{for} & t > n - \psi(i) \end{array}$$

Thus, this path is made up of "steps."

Since the path is 'flat', it is possible to compute the integrals and derive a simple formula for the SSM generated by the path $\gamma^{\psi,\hat{p}}$, denoted $x^{\psi,\hat{p}}$; however, this requires significant amounts of notation, so we only present the case when $n = 2$. In this case we compute

$$x_2^{(1,2),\hat{p}}(e; V) = V(e) - V(\hat{p}_1, e_2) + V(\hat{p}_1, 0)$$

and

$$x_2^{(1,2),\hat{p}}(e; V) = V(\hat{p}_1, e_2) - V(\hat{p}_1, 0)$$

which allows us to evaluate simple examples.

The almost flat method is made up as a "average" over all such paths:

$$x^{AF}(e;V) = \kappa^{-1} \sum_{\psi \in \Psi} \int_{[0,e]} x^{\psi,p}(e;V)dp$$

where $\kappa = |\Psi|\Pi_{i \in N} e_i$.

Once again, it is straightforward, but notationally complex, to compute the shares for the Almost Flat mechanism. For example when $V(p) = p_1^{\alpha} p_2^{\beta}$ for $\alpha, \beta \geq 0$ we see that

$$x_1^{AF}(e;V) = V(e)(\frac{\alpha}{\alpha+1} + \frac{1}{\beta+1})/2$$

and thus we can compute the shares for any polynomial V. Lastly note that this method is scale invariant.

The second mechanism that we consider is based on the weighted Aumann-Shapley mechanisms [19]. These are constructed from a weight vector, $w \in \Re^n_{++}$ and are generated by the paths $\gamma_i^{AS(w)}(t;e) = t^{w_i} e_i$ and are denoted by x^w. Now, the "Spread AS" mechanism is given by

$$x^{SPAS}(e;C) = \int_{(0,1]} x^{AS(w)}(e;C)dw.$$

This mechanism is also scale invariant.

Our third mechanism is a spread version of the Serial mechanism. For any weight vector, $w \in \Re^n_{++}$ and are generated by the paths $\gamma_i^{SER(w)}(t;e) = tw \wedge e$ and are denoted by x^w. Now, the "Spread Serial" mechanism is given by

$$x^{SPSER}(e;C) = \int_{(0,1]} x^{SER(w)}(e;C)dw.$$

This mechanism is demand monotonic.

By Lemmas 2 and 3 all three of these new mechanisms are strongly monotonic and have unique CTE's.

Theorem 5 *The following methods are strongly monotonic: Almost Flat, Spread Aumann-Shapley and Spread Serial methods. They also satisfy dummy and additivity. Two of them, Spread Aumann-Shapley and Spread Serial, have unique CTE's while the Almost Flat method has no CTE's.*

Corollary 1 *The is no SSM which satisfies additivity, dummy, strong monotonicity, scale invariance and demand monotonicity. However, there are many methods which satisfy additivity, dummy, strong monotonicity, and either scale invariance or demand monotonicity.*

5 Characterizations under Additivity and Roadblocks to Axiomatizations

As discussed in Section 2, under Additivity and Dummy, the set of SSM's are known to be integrals over path methods according to some measure, which we will denote by ν. Strong monotonicity and the existence of CTE's impose complementary constraints on this measure: intuitively (but *not* precisely) strong monotonicity essentially implies that the support of ν is dense, while CTE's exist only when ν has no atoms.

Since there are many measures which are atomless and have dense support, it is easy to construct methods which are both strongly monotonic and have CTE's. However, the characterization of any particular method is problematic as most known axiomatizations of

SSM's (see e.g., [5, 20, 9, 10]) either lead to a single SSM or a large family of them which is not easily parameterized as a subset of a finite Euclidean space. The author is unaware of any set of axioms that would single out a large but easily parameterized set of SSMs and believes that it is even more difficult to find a set of axioms which yield a specific (but nondegenerate) measure over such a class. This is a difficult problem which merits further study.

For example, consider the generalization of the Spread Aumann-Shapley method given by

$$x^{\omega}(e;C) = \int_{(0,1]} x^{AS(w)}(e;C)d\omega(w),$$

where ω is a measure on $[0,1]^n$. Even if we had a axiomatization of the set of weighted Aumann-Shapley methods, such an axiomatization would still leave many possible methods.[4] For any measure whose support is dense in $[0,1]^n$, this SSM will be strictly monotonic; however, it is difficult to find an axiom which will choose a specific, nondegenerate measure. One extreme possibility is the measure that puts probability $1/2^k$ on w^k, when $\{w^k\}_{k=0}^{\infty}$ is a dense subset of $[0,1]^n$; although such a measure would be ruled out if we also wanted to require that the SSM have a CTE, there are still many possible nondegenerate and atomless measures.

[4] While there exists axiomatizations of specific weighted Aumann-Shapley methods [19] there is no single axiomatization that yields the entire set. The closest result is in [9], which describes the set of scale invariant methods which contains the weighted Aumann-Shapley methods; however, this set is still too large to be parameterized simply as a subset of a Euclidian space.

6 Summary of Results

For easy reference we present the following table which summarizes our results, where columns represent methods and rows are for axioms. A check mark, $\surd$, indicates that the axiom is satisfied, while a dash, $-$, indicates that it is not.

	ES	PS	SS	AS	SER	AF	SPAS	SPSER
DUM	$-$	$-$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
ADD	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
SI	$\surd$	$\surd$	$\surd$	$\surd$	$-$	$\surd$	$\surd$	$-$
DM	$\surd$	$\surd$	$\surd$	$-$	$\surd$	$-$	$-$	$\surd$
MON	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
SMON	$\surd$	$\surd$	$-$	$-$	$-$	$\surd$	$\surd$	$\surd$
CDUM	$\surd$	$\surd$	$-$	$-$	$-$	$\surd$	$\surd$	$\surd$
CET	$\surd$	$\surd$	$-$	$-$	$-$	$-$	$\surd$	$\surd$

Thus, from consideration of the axioms introduced in this paper both the spread Serial and Spread Aumann-Shapley methods are attractive new cost sharing methods as they are the only methods considered which satisfy both dummy and all three new axioms; however, as discussed in the previous section, the set of CSMs which satisfy these is extremely large and apparently quite difficult to axiomatize.

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