Nash Bargaining without Scale Invariance

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Abstract

We present a characterization of the Nash Bargaining Solution on a domain which is not closed under Scale Invariance. The characterization on this restricted domain relies on Symmetry, Efficiency, Translation Invariance, the Independence of Irrelevant Alternatives and Continuity, but does not require Scale Invariance. Surprisingly, without continuity there exist a wide variety of bargaining solutions that satisfy the remaining axioms.

Keywords:

1. Introduction

Nash's characterization of the bargaining solution is one of the seminal results in game theory [11]. It initiated the field of axiomatic bargaining and the characterization of bargaining solutions. (See [17] for a survey of the area.) Nash's characterization relied on five fundamental axioms: Efficiency, Symmetry, Translation Invariance, Scale Invariance and the Independence of Irrelevant Alternatives (IIA).[1] The key idea in Nash's analysis is to transform a complex feasible region into a simpler region using the axioms. In particular, Nash's proof uses the IIA axiom to relate the solution of the given feasible region to one for a half-space and then uses Scale Invariance to relate this one to a solution of a simple symmetric bargaining problem. This is similar in spirit to many axiomatizations of bargaining solutions.

[1] Most presentations of the Nash Bargaining solutions, including Nash's, combine Translation Invariance and Scale Invariance into the single axiom of Affine Invariance; however, for our purposes we keep the two distinct.

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In this paper we demonstrate an alternative approach to the characterization of bargaining solutions. We consider an interesting domain which has been a subject of recent study [18, 1] in which the feasible regions are restricted to be polymatroids. This domain, which we discuss in the following section, is not closed under Scale Invariance and one can not use Nash's approach to characterize the solution in the usual manner; we provide a characterization which relies fundamentally on continuity. Somewhat surprisingly, there are many "reasonable" bargaining solutions which are not continuous, but satisfy the remaining axioms.

Polymatroids are interesting because they naturally capture allocation problems with complex constraints. They arise in many applications, such as congestion control on computer networks [13], rate allocation on wireless networks [14, 4, 10, 15], allocation of resources in data-centers and cloud computing [5, 3] and a variety of combinatorial problems arising from network constraints [1, 18]. Recent work [1, 3] shows that many bargaining solutions are particularly well behaved on polymatroids.

While many papers have considered larger and more complex domains (see [2, 19, 9]) ours considers the opposite, a simple subdomain, which has is well behaved mathematically[2], and as we discuss below it is surprising that the analysis is as rich as it appears to be.

Our analysis provides new insights for our understanding of the Nash Bargaining Solution as it differs significantly from others found in the literature for bargaining solutions. It also provides a case study for understanding restricted domains, on which there are fewer potential bargaining solutions (since many bargaining solutions which differ on larger domains may coincide on restricted domains) but also fewer restrictions imposed on these solutions, since there are not as many instances to compare. Lastly, it provides a characterization on an important domain of recent interest with practical applications.

Our characterization parallels that of Nash's, but our proof differs significantly. We assume Symmetry, Efficiency and Nash's Independence of Irrelevant Alternatives. However, we do not require Affine Invariance. In fact, much of Affine Invariance is essentially irrelevant as we discuss below. Surprisingly, we find that Continuity is a necessary assumption and show

[2]Polymatroids are defined by linear constraints where all coefficients are 0 or 1 and are simple to optimize over [6].

that there are many discontinuous bargaining solutions satisfying the remaining axioms. Our proof techniques, which are of independent interest, are based on a a pair of homotopies, an approach which may be of use in other bargaining problems.

2. Nash's Bargaining Solution

A bargaining problem consists of a set of players, N, a subset $A \subset \Re^N$, and a disagreement point $d \in A$. Given a domain $\mathcal{B}^N$, which is a collection of sets $A \subset \Re^N$ and and a disagreement point in each set, a bargaining solution is a mapping, $f(A, d) \in A$ with $f(A) \geq d$ for $d \in A$. In his paper, Nash considered the domain we denote $\mathcal{B}^N_{ccb}$ which consists of all nonempty subsets of $\Re^N$ which are convex closed and bounded above. Nash considered 5 axioms.

Given some $A \in \mathcal{B}^N$ the efficient frontier (assuming it is well defined for the domain $\mathcal{B}^N$) is denoted

$$EFF(A) = \{x \in A \mid \ \nexists y \in A, \ s.t. \ \ y \geq x \ \ \& \ \ y \neq x\}.$$

This leads to the first axiom.

Axiom 1. *Efficiency: For all $A \in \mathcal{B}^N$, $d \in A$, $f(A, d) \in EFF(A)$.*

Next, let $\pi : N \to N$ be a permutation of the players. The second axiom requires that the solution be invariant with respect to permutations.

Axiom 2. *Anonymity: For all $A \in \mathcal{B}^N$, $d \in A$ and permutations π on N, $\pi(f(A, d)) = f(\pi(A), \pi(d))$.*

Nash also considered the simple translations of coordinates. Let $\mu \in \Re^N$ and define T_μ to be the simple translation $T_\mu(x) = (\mu_1 + x_1, \mu_2 + x_2, \ldots, \mu_n + x_n)$. His third axiom is:

Axiom 3. *Translation Invariance: For all $A \in \mathcal{B}^N$, $d \in A$, and $\mu \in \Re^N$, $T_\mu(f(A, d)) = f(T_\mu(A), T_\mu(d))$.*

In addition he considered rescaling of coordinates. Let $\lambda \in \Re^N_{++}$ and define S_λ to be the scaling transformation $S_\lambda(x) = (\lambda_1 x_1, \lambda_2 x_2, \ldots, \lambda_n x_n)$. His fourth axiom is:

Axiom 4. *Scale Invariance: For all $A \in \mathcal{B}^N$, $d \in A$, and $\lambda \in \Re^N_{++}$, $S_\lambda(f(A, d)) = f(S_\lambda(A), S_\lambda(d))$.*

Note that in his original paper, an most of the literature, Translation Invariance and Scale Invariance are combined in a single axiom, Affine Invariance. For our purposes it is useful to distinguish between the two.

Nash's last, and most significant, axiom is based on the idea that eliminating unchosen outcomes should not change the solution.

Axiom 5. *Independence of Irrelevant Alternatives: If* $A, A' \in \mathcal{B}^N$, $d \in A$, $d' \in A'$ *such that* $A' \subset A$, $d = d'$ *and* $f(A, d) \in A'$ *then* $f(A', d') = f(A, d)$.

Now we state his main result.

Theorem 1 (Nash 1950). *There is a unique bargaining solution on* $\mathcal{B}^N_{ccb}$ *satisfying:*
(1) Efficiency
(2) Anonymity
(3) Translation Invariance
(4) Scale Invariance
(5) Independence of Irrelevant Alternatives.
This solution is denoted the Nash Bargaining Solution and for all $A \in \mathcal{B}^N_{ccb}$, $d \in A$, *the solution,* $\nu(A, d)$, *is the* $x \in A$ *that maximizes* $\Pi_{i \in N}(x_i - d_i)$ *(or equivalently: maximizes* $\Sigma_{i \in N} \log(x_i - d_i)$*).*

The key steps in Nash's proof, involve using the Independence of Irrelevant Alternatives to reduce the study of complicated bargaining sets to simple half-spaces and then using Affine Invariance to symmetrize the half-space so that the solution is obvious by Anonymity and Efficiency. Note that the Nash Bargaining Solution is Continuous in the data, so one does not need to add this explicitly to the axiomatization.

In the following, we characterize the Nash Bargaining Solution over an interesting subdomain on which these proof techniques can not be applied, since Scale Invariance is no longer relevant as, aside from uniformly scaling all dimensions simultaneously, there are no nontrivial bargaining sets which are related under these transformations. Thus, one of the main tools in Nash's analysis in not applicable.

As we will show, without Scale Invariance one must explicitly require continuity and apply a different and significantly more complex method of analysis using multiple homotopies, an approach which may prove useful in other bargaining problems.

3. Polymatroid Bargaining Sets

Now we reconsider Nash's analysis on an important subspace of $\mathcal{B}^N_{ccb}$, which has been a topic of recent interest and has many applications, the set of Polymatroids. To begin we recall the definition of a submodular function.

Given the set N, a function $C : 2^N \rightarrow \Re$ is submodular if for all $S, T \subseteq N$,

$$C(S \cup T) + C(S \cap T) \leq C(S) + C(T).$$

It is easy to check that one can translate a submodular function to attain a new submodular function, i.e. if $C(\cdot)$ is submodular and $x \in \Re^n$ then $C'(S) = C(S) - x(S)$ is also submodular, where $x(S) = \Sigma_{i \in S} x_i$.

The polymatroid[3] generated by the submodular function C is defined by

$$P(C) = \{x \in \Re^N \mid \forall\ S \subseteq N \ \ \sum_{i \in S} x_i \leq C(S)\}.$$

It is well known, and easy to check directly from the definitions, that the set of polymatroids are convex, closed and bounded above. Let $\mathcal{B}^N_P$ be the subset of $\mathcal{B}^N_{bcc}$ containing the polymatroids.

One interesting example of a polymatroidal bargaining set arise when one considers the allocation of divisible goods with mostly uniform valuations. For example, consider the allocation of foods among a set of players who's preferences are homogenious except for allergies. Thus, if two players are not allergic to a certain food, say peanut brittle, then they value it identically, while players with peanut allergies put no value on peanut brittle.

A related example, that arises in modern computing facilities, is the sharing of computers subject to constraints. For example, a modern cloud computing facility may have tens of thousands of computers to allocate among customers and the value of these computers is fairly homogeneous with a few strong exceptions, such as the operating system as software written for machines running Linux may not run properly on machines running Windows [5]. In this setting, the Nash Bargaining Solution is particularly interesting as it is both maximally fair, as described below, and strategyproof [3].

[3] Formally, we have defined the "extended polymatroid" while the standard polymatroid is attained by taking the intersection of the extended polymatroid with the non-negative quadrant $\Re^N_+$ [6]. Our results could be extended to standard polymatroids by applying the IIA axiom.

These examples can be modeled by set games, where there is a finite set of divisible goods G and each player $i \in N$ has nonzero valuations for a subset of the goods, $S_i \subset G$, and these sets can be chosen arbitrarily; however, the valuation of each good is uniform *among the players that desire it,* and additive. Let x_i^g be the fraction of good g that is allocated to player i. Then, there exists a representation of the players' utilities and a function $V : G \to \Re$ such that for any allocation x,

$$U_i(x) = \sum_{g \in S_i} V(g).$$

For example, consider the case of 2 players and 3 goods, where $V(g) = 1$ for all three goods with $S_1 = \{1, 3\}$ and $S_2 = \{2, 3\}$, i.e. player one desires good 1 while player 2 desires good 2, and both desire good 3. For this model, the feasible allocations are simply $x_1 = (\alpha, \beta, \gamma)$ and $x_2 = (1-\alpha, 1-\beta, 1-\gamma)$ for any $x_1 \in [0, 1]^3$. The bargaining set is given by

$$\{(u_1, u_2) \in \Re^2_+ \mid u_1 \leq 2, \quad u_2 \leq 2, \quad u_1 + u_2 \leq 3\},$$

where the relevant set function is given by

$$C(\{1\}) = 2,$$

$$C(\{2\}) = 2,$$

$$C(\{1, 2\}) = 3,$$

which can be directly checked to be submodular.

For completeness, we note that set games $< N, G, S, V >$ generate bargaining sets which are polymatroids. In addition, many other more complex games also generate bargaining sets which are polymatroids. See [18] for more examples and [3] which shows that the Nash Bargaining Solution is group strategyproof in set games.

Also, note that set games arise naturally when one views cooperative games with transferable utility in a bargaining framework.

4. Bargaining on Polymatroids

The Nash Bargaining Solution is well defined on $\mathcal{B}_P^N$ and satisfies all five axioms from Nash's theorem, since $\mathcal{B}_P^N \subset \mathcal{B}_{ccb}^N$. However, it is not necessarily true that on this restricted domain the Nash Bargaining Solution is the only bargaining solution that satisfies these axioms. Initially, one might suppose that this is true, since the set of bargaining solutions is reduced on a restricted domain, as different solutions on $\mathcal{B}^N_{ccp}$ coincide on the more restrictive domain $\mathcal{B}^N_P$, as we will see below and also in [16]. Somewhat surprisingly, it is not. This arises due to a counterpoint, that on more restricted domains there are fewer points of comparison (bargaining sets) for axioms to apply to. For example, on $\mathcal{B}^N_P$ Scale Invariance is extremely weak, since for most values of $\lambda \in \Re^N_+$ and most submodular functions C, the transformed polymatroid $S_\lambda(P(S))$ is not a polymatroid. This is because polymatroids are defined by constraints of the form:

$$\sum_{i \in N} u_i \leq C(S)$$

which is altered under S_λ to

$$\sum_{i \in N} \lambda_i u_i \leq C(S)$$

which can not arise in a polytope unless all λ_i are identical (or the constraint is never binding). Thus, we only need to consider uniform changes of scale, in which all λ_i are identical.

First we show that Nash's axioms are not sufficient to characterize a bargaining solution over the polymatroids as Scale Invariance does not provide enough restrictions.

Set $N = \{1, 2\}$ and assume that $d = 0$ and $C(N) = 1$. The set of all such polymatroids is parametrized by two parameters: $C(\{1\}) = \alpha$ and $C(\{2\}) = \beta$ for $\alpha, \beta > 0$ (since $d = 0$) where $C(\{1, 2\}) = 1$. The requirement that C be submodular implies that $\alpha + \beta \geq 1$ and we can assume w.l.o.g. that $\alpha, \beta \leq 1$. Thus, we can think of the bargaining solution on this set of polytopes as a function of two variables, α, β.

We now construct a bargaining solution f that differs from the Nash Bargaining solution g. First we consider conditions under which it agrees with the Nash Bargaining Solution. If both $\alpha \geq 1/2$ and $\beta \geq 1/2$ then

$$f(\alpha, \beta) = g(\alpha, \beta) = (1/2, 1/2).$$

(This must be so, by Independence of Irrelevant Alternatives.) In addition if

$$1 > \alpha > 1/2 > \beta \geq 0$$

set

$$f(\alpha, \beta) = g(\alpha, \beta) = (1 - \beta, \beta).$$

Similarly if

$$1 > \beta > 1/2 > \alpha \geq 0$$

set

$$f(\alpha, \beta) = g(\alpha, \beta) = (\alpha, 1 - \alpha).$$

Now we consider conditions where it differs from the Nash Bargaining Solution. When $\alpha = 1$ and $\beta < 1/2$ set

$$f(\alpha, \beta) = (1, 0) \neq g(\alpha, \beta),$$

and when $\beta = 1$ and $\alpha < 1/2$ set

$$f(\alpha, \beta) = (0, 1) \neq g(\alpha, \beta).$$

Note that f is clearly discontinuous.

We extend this to a bargaining solution over the full domain by Translation Invariance (for $d \neq 0$) and Scale Invariance, with all λ_i's identical (for $C(N) \neq 1$). Then it is obvious, by construction, that f satisfies Efficiency, Anonymity, Translation Invariance and Scale invariance. To see that it satisfies Independence of Irrelevent Attributes consider some polytope defined by α, β. We must show that for any $\alpha' \leq \alpha$ and $\beta' \leq \beta$ if $f(\alpha, \beta) \in P(\alpha', \beta')$ then $f(\alpha', \beta') = f(\alpha, \beta)$. Since f and g agree when $\alpha < 1$ or $\beta < 1$ we need only consider the case when $\alpha' = 1$ or $\beta' = 1$, since in the other cases we are comparing the Nash Bargaining Solution with itself, so we know it satisfies Independence of Irrelevant Alternatives.

Now consider the case when $\alpha = 1$ and $\beta < 1/2$. If $\alpha' = 1$ and $\beta' \leq \beta$ then

$$f(\alpha', \beta') = f(\alpha, \beta),$$

while if $\alpha' < 1$ and $\beta' \leq \beta$,

$$f(\alpha, \beta) = (1, 0) \notin P(\alpha', \beta'),$$

showing that Independence of Irrelevant Alternatives is satisfied, completing the proof for $N = \{1, 2\}$.

This example can be extended to construct many other solutions that satisfy Nash's axioms on the polymatroids. For example, one can require that $\alpha = 1$ and $\beta < \gamma$ for any $\gamma < 1/2$ before deviating from the Nash Bargaining Solution to create a whole family of bargaining solutions and even more complex bargaining solutions can be attained which have infinitely many points of discontinuity. In addition, for arbitrary N one can parallel the construction for $N = \{1, 2\}$ to create other bargaining solutions. Perhaps the simplest such solution agrees with the Nash Bargaining solution everywhere except when there exists some $i \in N$ such that $C(\{j\}) < 1/n$ for all $j \neq i$ and $u = e_i$ is in the bargaining set, in which case $f(C) = e_i$.

The non-Nash bargaining solutions that we constructed are all discontinuous. As we will show below, when restricted to continuous solutions we retain the desired characterization. The simplest way to define continuity in our setting is to view C as a point in $\Re^{2^N}$ and then require that the bargaining solution, as a function from $\Re^{2^N}$ to $\Re^N$, is continuous in the standard topology on $\Re^N$. We phrase this in a way which will be convenient for our analysis.

Axiom 6. *Continuity: Given a sequence of submodular functions C^k and disagreement points $d^k \in EP(C^k)$ such that*

$$\lim_{k\to\infty} C^k = C$$

and

$$\lim_{k\to\infty} d^k = d,$$

then

$$\lim_{k\to\infty} f(C^k, d^k) = f(C, d).$$

To state our main theorem we remove the unnecessary part of Scale Invariance and add Continuity.

Axiom 7. *Uniform Scale Invariance: For all $A \in \mathcal{B}^N$, $d \in A$, and $\lambda \in \Re^N_{++}$ with $\lambda_i = \lambda_j$ for all $i, j \in N$,*

$$S_\lambda(f(A, d)) = f(S_\lambda(A), S_\lambda(d)).$$

Theorem 2. *The Nash Bargaining Solution is the unique bargaining solution on $\mathcal{B}_P^N$ satisfying:*
(1) Efficiency
(2) Anonymity

(3) Translation Invariance
(4) Uniform Scale Invariance
(5) Independence of Irrelevant Alternatives.
(6) Continuity.

First we note that w.l.o.g. we can set $d = 0$ by a simple translation as polymatroids are invariant under translations. (Simply define $C'(s) = C(S) - \sum_{i \in S} d_i$.) In the following we will assume that $d = 0$ throughout.

Before proving this theorem we present a useful result discussed in [1]. It provides a simple, and somewhat surprising, characterization of the Nash Bargaining Solution on extended polymatroids in terms of lexicographic max-min ordering, which is well known from the social choice theory propounded by Rawls [12]. Given a vector, $r \in \Re^N$, let $L(r)$ be the permutation of the elements of r such that

$$L_1(r) \leq L_2(r) \leq \cdots \leq L_n(r).$$

We say that $r \in \Re^N$ is larger than $r' \in \Re^N$ according to the lexicographic max-min ordering if $L(r)$ is lexicographically larger than $L(r')$, i.e. there exists some $1 \leq i \leq n$ such that $L_i(r) > L_i(r')$ and $L_j(r) = L_j(r')$ for all $j < i$.

Theorem 3 ([1]). *The Nash Bargaining Solution on a polymatroid maximizes the lexicographic max-min on the feasible region.*

One interesting connection is that the lexicographic max-min ordering, after normalization, arises in the natural extension of the Kalai-Smorodinsky Bargaining Solution [8] for more than two players as shown in [7].

The first step in our proof is the analysis of a simple algorithm for computing the lexicographic max-min on a polymatroid. This is essentially the same procedure used by Imai for the Lexicographic Kalai-Smorodinsky solution [7]. Begin with $k = 0$, $X(0) = 0 \in \Re^N$, $B(0) = \emptyset$, and $T(0) = 0$ and proceed recursively. Define $T(k+1)$ to be the infimum of $t > T(k)$ such that

$$x(t) = X(k) + (t - T(k))e_{N \setminus B(k)}$$

is infeasible. As we will show below, there is a well defined maximal set, $B(k+1) \supset B(k)$ which is binding,

$$\sum_{i \in B(k+1)} x_i(T(k+1)) = C(B(k+1)).$$

We then set

$$X(k+1) = x(T(k+1))$$

and repeat with

$$k \to k+1$$

until $B(k) = N$ in which case we define k^* to be the terminating value of k.

Lemma 1. *The above procedure for computing the lexicographic max-min is well defined and correct.*

Proof: The correctness of this procedure is clear, since it directly computes the lexicographic max-min once we show that the minimal subset step is well defined. Suppose that there were two maximal subsets at iteration k. Call them $S \neq S' \subset N$ and assume that neither $S \subseteq S'$ nor $S' \subseteq S$. By assumption

$$\sum_{i \in S} x_i(T(k+1)) = C(S)$$

which can be written as

$$\sum_{i \in B(k)} x_i(T(k)) + \sum_{i \in S \setminus B(k)} T(k+1) = C(S)$$

and a similar equation for S'

$$\sum_{i \in B(k)} x_i(T(k)) + \sum_{i \in S' \setminus B(k)} T(k+1) = C(S')$$

Now consider the related constraint for the nonbinding on the set $S \cup S'$

$$\sum_{i \in B(k)} x_i(T(k)) + \sum_{i \in (S \cup S') \setminus B(k)} T(k+1) < C(S \cup S')$$

and the related constraint for the binding or nonbinding on $S \cap S'$

$$\sum_{i \in B(k)} x_i(T(k)) + \sum_{i \in (S \cap S') \setminus B(k)} T(k+1) \leq C(S \cap S')$$

Recall the submodularity constraint

$$C(S) + C(S') \geq C(S \cup S') + C(S \cap S')$$

and by substituting into the left hand side of this equation from the left hand sides of the previous equations and removing the common terms we get

$$\sum_{i \in S \setminus B(k)} T(k+1) + \sum_{i \in S' \setminus B(k)} T(k+1) \geq C(S \cup S') + C(S \cap S')$$

while repeating this procedure for the right hand side of the submodularity equation yields

$$C(S) + C(S') > \sum_{i \in (S \cup S') \setminus B(k)} T(k+1) + \sum_{i \in (S \cap S') \setminus B(k)} T(k+1).$$

Combining these and dividing by $T(k+1)$ yields

$$|S \setminus B(k)| + |S' \setminus B(k)| > |(S \cup S') \setminus B(k)| + |(S \cap S') \setminus B(k)|,$$

but simple counting shows that the two sides of this inequality are actually equal, providing a contradiction and completing the proof. □

In the following, we will focus on the maximal binding inequalities used in the above procedure, which we identify by the sets

$$B = \{B(1), B(2), \ldots, B(k^*)\},$$

where $B(k^*) = N$. Given a submodular function, $C(\cdot)$, we define the submodular function inflated by the telescoping sequence A as follows. Let

$$K(S) = \max\{k \mid S \cap (N \setminus B(k)) \neq \emptyset\}$$

and then define $C_B(S) = C(B(K(S)))$.

Lemma 2. *For any telescoping sequence*

$$B = \{B(1), B(2), \ldots, B(k^*) = N\}$$

derived from the Lexicographic max-min procedure defined above,

$$C(B(1)) < C(B(2)) \cdots < C(B(k^*))$$

and $C_B(\cdot)$ is submodular.

Proof: It is easy to see that

$$C(B(1) < C(B(2)) \cdots < C(B(k^*))$$

from the construction since

$$\sum_{i \in B(k)} x_i(t) = C(B(k)).$$

For the second claim, consider $S, S' \subseteq N$ and the required inequality for submodularity

$$C_B(S) + C_B(S') \geq C_B(S \cup S') + C_B(S \cap S').$$

First note that if $K(S) = K(S')$ then $K(S \cup S') = K(S)$ while $K(S \cap S') \leq K(S)$ so

$$C_B(S) + C_B(S') = 2C(B(K(S)))$$

while

$$C_B(S \cup S') + C_B(\cap S') \geq C(B(K(S))) + C(B(K(S)))$$

combining these provides the correct inequality. Next assume that $K(S) \supset K(S')$. Then $K(S \cup S') = K(S)$ while $K(S \cap S') \subseteq K(S')$ showing that

$$C_B(S) + C_B(S') = C(B(K(S))) + C(B(K(S')))$$

while

$$C_B(S \cup S') + C_B(\cap S') \geq C(B(K(S))) + C(B(K(S')))$$

and combining these two provide the correct inequality and completes the proof. □

Next we show that this inflated submodular function induces a relaxation of the polymatroid.

Lemma 3. *Suppose that B is generated by the Lexicographic max-min procedure on $P(C)$ then $P(C) \subseteq P(C_B)$.*

Proof: Suppose that $x \in P(S)$ but there exists some $S \subseteq N$ such that

$$\sum_{i \in S} x_i > C_B(S).$$

By construction $S \subset B(K(S))$ and since $x \in P(C)$

$$\sum_{i \in B(K(S))} x_i \leq C(B(K(S))).$$

Now $C(B(K(S))) = C_B(B(K(S)))$ so

$$\sum_{i \in B(K(S))} x_i \leq C_B(B(K(S)))$$

which implies that

$$\sum_{i \in S} x_i \leq C_B(B(K(S))) = C_B(S)$$

providing a contradiction and proving the lemma. □

This implies that we can restrict our attention to the inflated polymatroids since:

Lemma 4. *If a bargaining solution f satisfies IIA then $f(C_B) = f(C)$ for any B that is generated by the Lexicographic max-min procedure on $P(C)$.*

Proof: This follows from the previous lemma since $P(C) \subseteq P(C_B)$. □

This suggests that we consider a subset of the polymatroids which encompasses these inflated polymatroids. Given $k^* > 0$, a telescoping sequence of sets

$$B = \{B(1), B(2), \ldots, B(k^*) = N\}$$

and a function $D(k)$ such that

$$D(1) \leq D(2) \leq \cdots \leq D(k^*)$$

define the "inflated" polytope $F(k^*, B, D)$ from the submodular function $C^{k^*,B,D}$ as above, i.e. set

$$K(S) = \max\{k \mid S \cap (N \setminus B(k)) \neq \emptyset\}$$

and define $C^{k^*,B,D}(S) = D(K(S))$. Let $\mathcal{B}_F^N \subset B_P^N$ be the set of all such polymatroids. The above lemma implies that we only need to prove the characterization on $\mathcal{B}_F^N$ to complete the proof.

Lemma 5. *The Nash Bargaining Solution is the unique bargaining solution on $\mathcal{B}_F^N$ satisfying Efficiency, Anonymity, Continuity and Independence of Irrelevant Alternatives.*

Proof: Consider $F(k^*, B, D) \in \mathcal{B}_F^N$ for arbitrary k^*, B, D, satisfying the above requirements and set $D(k^*) = n$ by Uniform Scale Invariance. Now define a homotopy from a simple symmetric submodular function to D defined by

$$D^s(S) = (1-s)D(S) + s|S|.$$

Let f be a bargaining solution satisfying the assumptions of the theorem and define

$$P^s = F(k^*, B, D^s)$$

and $f^s = f(P^s)$. Let $x^s = \nu(P^s)$ where ν is the Nash Bargaining Solution and note that

$$x^s = (1-s)x^1 + se$$

since this satisfies all the constraints for the sets in B and would be constructed by the lexicographic max-min procedure.

In addition to the analysis of the homotopy we also use induction on k^*. Note that for the base case, $k^* = 1$, it is easy to see by Efficiency and Anonymity that $x^1 = eD(1)/n = \nu(P^1)$. So we will inductively assume that

$$f(F(k, B, D)) = \nu(F(k, B, D))$$

for all $k < k^*$ and B, D.

Next note that, by continuity, f^s is continuous in s and consider the "path" of f^s for $s \in [0, 1]$. For contradiction assume that $f^1 \neq \nu(C^1)$. Since $\nu(C^1)$ is constructed by requiring that all k^* constraints

$$\sum_{i \in B(k)} f_i^s = C^s(B(k)) \quad \forall 1 \leq k \leq k^*$$

are tight, this implies that for at least one $1 \leq k \leq k^*$, say k', the above constraint is not tight. In particular

$$\sum_{i \in B(k')} f_i^s = \phi < C^s(B(k')).$$

Now consider a different homotopy where

$$D^r(k') = r$$

and $D^r(k) = D(k)$ for all $k \neq k'$ and $r \in [D(k), D(k+1)]$, i.e. we vary only one component of D. Note that for $P(F(k^*, B, D^r))$ when $r = D(k'+1)$ the k' constraint is irrelevant, since it never binds unless the $k'+1$'st constraint does. Thus, for $r = D(k'+1)$, $P(F(k^*, B, D^r))$ can be generated by a B with $k^* - 1$ sets, B with $B(k)$ removed, and by the inductive hypothesis

$$f(P(F(k^*, B, D^r))) = \nu(P(F(k^*, B, D^r))).$$

Also note that when $r = D(k')$, $D^r = D$.

Consider the function

$$g(r) = \sum_{i \in B(k')} f_i^s(P(F(k^*, B, D^r)))$$

and recall that for $r = D(k')$,

$$g(r) = \phi < D(k')$$

while for $r = D(k'+1)$,

$$g(r) = D(k'+1).$$

By the continuity of f, $g(r)$ must also be continuous so there exists some

$$\tilde{r} \in (D(k'), D(k'+1))$$

with

$$g(\tilde{r}) = D(k').$$

Then, by IIA,

$$g(D(k')) = g(\tilde{r}) = D(k'),$$

since the solution remains feasible when r goes from $\tilde{r}$ to $D(k')$, but by assumption

$$g(D(k')) = \phi < g(D(k'))$$

creating a contradiction and proving the lemma. □

Thus, we have show by the continuity of the homotopy that the axioms uniquely characterize the Nash Bargaining Solution on $\mathcal{B}_F^N$ and by the previous lemma this completes the proof of our characterization of the Nash Bargaining Solution on the polymatroids.

5. Conclusions

We have shown that the characterization for the Nash Bargaining Solution fails on a interesting domain that does not satisfy Scale Invariance, since there are many discontinuous bargaining solutions satisfying his axioms. However, the Nash Bargaining Solution is the only continuous solution on this domain, but the proof of this fact requires a different approach than is typical in the analysis of Bargaining solutions and provides new insights and methods of analysis for the study of bargaining solutions.

There remain many interesting open questions and issues raised by this analysis. First, there are other challenging problems in bargaining theory where the homotopy based analysis used in this paper may be applicable. Second, there are many possible extension of this analysis to other interesting domains. For example, is the restriction to submodular functions to generate the polymatroid necessary? Chackrobaty et. al. [1] consider feasible regions which are described by the intersection of halfspaces with binary normal vectors, but which are not polymatroids. In this case, the Nash Bargaining Solution does not coincide with lexicographic max-min solution so our analysis does not carry over directly, although most of our homotopy analysis does, suggesting that a similar characterization might hold in that setting.

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