Paths and Consistency in Additive Cost Sharing

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Abstract

We provide a direct proof of a representation theorem for additive cost sharing methods as sums of path methods. Using this representation we show that many of the standard additive cost sharing methods (Aumann-Shapley, Shapley Shubik, and Serial Cost) are consistent. These results follow directly from a simple sufficient condition for consistency: being generated by associative paths, which can be used to show consistency for many other methods.

We introduce a new axiom, dummy consistency, which is quite mild. Nonetheless there is an important relationship between dummy consistency and consistency.

Using dummy consistency, we also show that the Aumann-Shapley and Serial Cost methods are the *unique* (additive) consistent extension of their restriction on all two agent problems, while the Shapley-Shubik method has multiple consistent extensions but a unique symmetric one. In fact, these results are unchanged when we replace consistency with dummy consistency.

1 Introduction

The principle of consistency has a long history in many economic allocation problems. (See [26] for a detailed review and bibliography.) For example, under standard assumptions, the Walrasian allocation is consistent in the appropriate sense and indeed can be characterized

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by consistency in combination with other basic axioms. The basic idea behind consistency in any allocation problem is that, if an agent leaves the problem and takes her share with her, then the allocation among the remaining agents is unchanged.

Consistency is also a key property studied in cooperative game theory.[1] Since any cost sharing mechansism is mathematically equivalent to the value of a cooperative game, such analyses can often be directly translated into our context, which is the study of cost sharing methods, although interpretations often differ in the two contexts.[2]

For example, Hart and Mas-Colell [14] showed, using a standard notion of consistency, that the Shapley value is the unique consistent extension which "divides the surplus equally" in 2 person games, using a new tool, the potential of a game. Mclean, Pazgal and Sharkey [15] extended this result to the case with continuous demands, showing that the Aumann-Shapley method is the unique consistent extension of the Aumann-Shapley method defined on all 2 agent problems. Their proof of this result is quite involved and also uses a notion of potential.

In this paper we show that, when restricted to additive methods, the existence and uniqueness of consistent extensions can be analyzed much more directly. First we provide a sufficient condition for a CSM to be consistent. Using this we show that the Aumann-Shapley, Shapley-Shubik and Serial Cost methods are all consistent as are the weighted, asymmetric, versions of all of these.

Next we introduce an extremely weak version of consistency, dummy consistency, and

[1]For a recent review of consistency in cooperative games see [5] and the references therein.

[2]For a general introduction to cost sharing see e.g., [29] and for details of the approach taken in this paper see [9].

provide a sufficient condition for a CSM to be dummy consistent. Dummy consistency is an extremely natural requirement and thus may be applicable in a wider variety of situations than ordinary consistency. Nonetheless, in many cases the existence and uniqueness of dummy-consistent extensions implies the existence and uniqueness of consistent extensions. This allows us to easily prove the existence and uniqueness of consistent extensions from 2 agent problems to multi-agent problems for the Aumann- Shapley and Serial Cost methods, while the Shapley-Shubik method may have multiple extensions, but a unique symmetric one.[3]

Our results and method of analysis are based on a representation for additive CSMs based on the idea of a path method, which is a natural generalization of the Aumann-Shapley formula. This representation method is also of independent interest for other applications (see, e.g., [8, 6]). The first complete proof of this result was given by Haimanko [11] in the context of non-atomic games.[4] In this paper we provide a direct proof of this result in the cost sharing context, which may be of independent interest. Our proof relies upon an analogous theorem in the discrete context by Wang [27].

The paper is organized as follows: Section 2 reviews the structure of the class of additive CSMs and describes their representation as sums of path methods, the key technique used in this paper. Section 3 provides sufficient conditions for a CSM to be consistent and a dummy-consistent. We conclude in Section 4 with the analysis of consistent extensions. Several proofs are in the appendix.

[3]These results can also be used to analyze asymmetric CSMs

[4]An earlier version of this paper stated this result; however, the proof in that version was incorrect.

2 Additive Cost Sharing

We will consider the problem of dividing the cost of production among a group of agents, where each agent has a demand for an idiosyncratic good. Thus, we will identify agents with their good, and since we will be interested in the consistency of various methods we will allow for the set of agents to vary.

Let the set of potential agents be identified with the set of nonnegative integers, Z_+, and let $\mathcal{N}$ be the set of nonempty finite subsets of Z_+. For some finite set of agents $N \in \mathcal{N}$, their allowed demands will be denoted by $q \in Q(N, \overline{q}) = [0, \overline{q}]^N$, where often $\overline{q} = \infty$ which implies that $Q(N, \overline{q}) = [0, \infty)^N$. To save on notation, we will think of $\overline{q}$ as fixed and therefore it will only be implicit in the notation. For example we will usually write $Q(N)$ instead of $Q(N, \overline{q})$. The cost of serving these demands is $C(q)$ with $C \in \mathcal{C}(N)$, where $\mathcal{C}(N)$ is the set of nondecreasing, continuously differentiable functions from $Q(N)$ to $\Re_+$, satisfying $C(0) = 0$.[5] A cost sharing mechanism provides a method for computing the cost shares allocated to each of the agents. The following definition is introduced in [9].

Definition 1 *Given $N \in \mathcal{N}$, an additive cost sharing method x is a mapping $x : Q(N) \times \mathcal{C}(N) \to \Re_+^N$ satisfying:*

1) Efficiency: $\sum_{n \in N} x_n(q; C) = C(q)$,

2) Additivity: for all $C, D \in \mathcal{C}(N)$ the following holds: $\forall n \in N, \quad x_n(q; C + D) = x_n(q; C) + x_n(q; D)$.

3) Dummy: For any $C \in \mathcal{C}(N)$ such that $\partial_n C(p) = 0$ for all $p \in Q(N)$, then the following

[5] Note that the generic vector of 0's will be denoted by 0 and the unit vector of 1's by 1, where the dimension of these vectors will be obvious from the context.

must hold: $x_n(q;C) = 0$ *for all* $q \in Q(N)$.

Let $CS(N)$ *denote the set of all such CSMs.*

Note that we use the notation $\partial_n C(p)$ to represent the partial derivative of $C(q)$ with respect to q_n evaluated at p. Let $CS(q,N)$ be the projection of $CS(N)$ onto its second component, i.e., if $x(\cdot;\cdot) \in CS(N)$ then $x(q;\cdot) \in CS(q,N)$. Also let $CS = \bigcup_{N \in \mathcal{N}} CS(N)$. The following representation theorem is given in [9].

Proposition 1 (Friedman and Moulin, 1998) *For any* $N \in \mathcal{N}$ *and* $x \in CS(q,N)$*, there exists a vector of nonnegative Radon measures indexed by* q*,* $\{\rho_n(q)\}_{n \in N}$ *each defined on* $[0,q]$ *such that for all* $q \in Q(N)$*,* $C \in C(N)$ *and* $n \in N$*:*

$$x_n(q;C) = \int_{[0,q]} \partial_n C(p) d\rho_n(q)(p)$$

and for each $q \in Q(N)$ *and* $n \in N$ *the projection of* $\rho_n(q)$ *onto the* q_n *axis is the Lebesgue measure, i.e.,* $\rho_n(q)([a,b] \times [0,q_{-n}]) = b - a$ *for* $0 \leq a \leq b \leq q_n$.

Note that, as written, this is not a complete characterization, since not all vectors of measures lead to valid CSMs. In particular, budget balance need not be satisfied. In [9], a set of restrictions on the measures was given which leads to a proper characterization theorem; however, those conditions are quite complicated. We now present a second representation theorem that is much more straightforward and intuitive. It is based on an important class of CSMs, the path methods.

Definition 2 *A path function* γ *is a mapping* $\gamma : [0,\infty] \times Q(N) \to Q(N)$ *satisfying the following for each* $q \in Q(N)$*:*

1) $\gamma(t;q)$ is continuous and nondecreasing in t.

2) $\gamma(0;q)=0$ and there exists a $\hat{t}>0$ such that for all $t\geq\hat{t}$, $\gamma(t;q)=q$.

Let the set of all such path functions be denoted $\Gamma(N)$.

Also, for each $q\in Q(N)$ let $\Gamma(q,N)$ be the projection of $\Gamma(N)$ onto its second component, for fixed q. Given such a path function, it is straightforward to construct a related CSM.

Definition 3 *Given a path $\gamma\in\Gamma(N)$ the "path method generated by γ" is given by*

$$x_n^{\gamma}(q;C)=\int_{t=0}^{\infty}\partial_n C(\gamma(t;q))d\gamma_i(t;q).$$

As proven in [9], a path method is a valid CSM. A path method is the natural generalization of the well known formula for the Aumann-Shapley method (discussed in the following section). Note also that any convex combination of two CSMs is a valid CSM, and thus any convex combination of path methods is also a CSM. In fact, as the following theorem demonstrates, any CSM can be constructed as a convex combination of path methods.

Theorem 1 *For any $N\in\mathcal{N}$ the following are equivalent:*

i) $x\in CS(N)$.

ii) There exists a family of probability measures, indexed by $q\in Q(N)$, μ^q, each on $\Gamma(q,N)$ such that

$$x=\int_{\gamma(\cdot;q)\in\Gamma(q,N)}x^{\gamma(\cdot;q)}d\mu^q(\gamma(\cdot;q)).$$

The proof is given in the appendix.

As mentioned in the introduction, the first complete proof of this result was given by Haimanko [11] in the context of nonatomic games. (See also, Haimanko [12].) In that setting, one can view each (infinitesimal) agent, in the cooperative game, as an infinitesimal unit of a good. The translation of the axioms in that paper can be easily translated in the basic axioms used in this paper. Nonetheless, we believe that a "direct proof" in the cost sharing context may be of independent interest.

2.1 Examples of path methods

One of the best known CSMs is the Aumann-Shapley method. This method was first applied in Billera, Heath and Ranaan[1] to allocate the cost of a shared telephone system and was subsequently characterized in terms of economic axioms in Billera and Heath [2] and Mirman and Tauman [17]. For any $N \in \mathcal{N}$, it is a path method, x^{AS}, generated by the path $\gamma_n(t;q) = \min[tq_n, q_n]$, which is the well known "diagonal path."

A set of nonsymmetric variants of this method has been proposed by Mclean and Sharkey [16]. These are constructed with a weight function $w : N \to \Re_+ \setminus \{0\}$ and are the path methods given by $\gamma_n(t;q) = \min[t^{w(n)}q_n, q_n]$.

Another well known CSM was proposed by Shubik [24], and is known as the Shapley-Shubik method, since it is the Shapley value of the cooperative game generated from the cost function. Before constructing this method we first consider the class of incremental methods. An incremental method is constructed from an ordering $\omega \in \Omega(N)$, where $\Omega(N)$ is the set of all orderings, i.e., bijective functions $\omega : N \to (1, 2, \ldots, |N|)$. The incremental method for order ω is computed as follows.

For any $n \in \mathcal{N}$ and $q \in Q(N, \infty)$ define $S^+(n;\omega) = \{m \in N \mid \omega(m) \leq \omega(n)\}$ and $S^-(n;\omega) = S^+(n;\omega) \setminus n$ then $x_n^\omega(q;C) = C(q_{S^+(n;\omega)}, 0_{-S^+(n;\omega)}) - C(q_{S^-(n;\omega)}, 0_{-S^-(n;\omega)})$. This is equivalent to the path method generated by the function $\gamma_n^{\omega}(t;q)$ which is 0 for $t < \omega(n)-1$, q_i for $t > \omega(n)$ and $(t-\omega(n))q_i$ otherwise.

We can generalize this to a random order method [28] by choosing a set of weights on each ordering $w : \Omega(N) \to \Re_+ \setminus \{0\}$ which must satisfy $\sum_{\omega\in\Omega(N)} w(\omega) = 1$. Then the random order method with weights w is given by $x^w = \sum_{\omega\in\Omega(N)} w(\omega)x^{\omega}$. The Shapley-Shubik method is given by the Random Order method with equal weights:

$$x^{SS} = (|N|!)^{-1} \sum_{\omega\in\Omega(N)} x^{\omega}.$$

The generalized serial cost method was introduced in [9] and is based on the serial cost method for homogeneous goods which was analyzed in [21]. The serial cost method for homogeneous goods which was motivated by Fair Queuing [4] in the networking literature and is interesting both axiomatically and strategically.[6] The serial cost method, x^{SC} is generated by the path $\gamma_n(t;q) = \min[t, q_n]$.

Given a weight function $w : P \to \Re \setminus \{0\}$ we can define the linearly weighted serial cost method as the CSM generated by the path $\gamma_n(t;q) = \min[w(n)t, q_n]$ and the exponentially weighted serial cost method as the method generated by the path $\gamma_n(t;q) = \min[t^{w(n)}, q_n]$, two new classes of methods which are applied in [8].

2.2 Scale Invariance

So far, we have imposed no restrictions on the relationship of cost shares for different demands, i.e., given a CSM x and a cost function C we have imposed no restrictions on the

[6] Shenker [23], and Moulin and Shenker [21], have shown that games induced by serial cost are dominance solvable, while Friedman and Shenker [10] have shown that such games are also solvable in overwhelmed actions, and thus are robustly learnable for a wide class of learning algorithms and information structures. The generalized serial cost method [9] has similar axiomatic and strategic characterizations [7].

relationship between $x(q;C)$ and $x(q';C)$ even if q and q' are very close. For example, one might want to require that $x(q;C)$ be continuous in q. Scale Invariance, which we now discuss, is stronger than continuity.

Scale invariance, is well known and was used in the original axiomatizations of the Aumann-Shapley method. Given $\lambda \in \Re^N_{++}$, define $\tau_\lambda(q)$ by $\tau_\lambda(q)_n = \lambda_n q_n$ for $n \in N$ and define $\tau_\lambda(C)$ by $\tau_\lambda(C)(q) = C(\tau_\lambda(q))$, for $C \in \mathcal{C}(N,\infty)$.

Definition 4 (Scale Invariance) *For all $N \in \mathcal{N}$ a CSM, $x \in CS(N,\infty)$, is scale invariant if $x(\tau_\lambda(q);C) = x(q;\tau_\lambda(C))$, for all $\lambda \in \Re^N_{++}$ and $C \in \mathcal{C}(N,\infty)$.*

Scale invariance is the statement that different goods are not comparable, e.g. the units used to measure q_n cannot be meaningfully compared with those used to measure q_m when $n \neq m$. We now show that the set of scale invariant CSMs are generated by the scale invariant paths. First however, we need a weak version of continuity, continuity at 0, to simplify the presentation.

Definition 5 *Given any $N \in \mathcal{N}$, a CSM $x \in CS(N)$ is* continuous at 0[7] *if for all $C \in \mathcal{C}(N)$, $n,m \in N$ and $q \in Q(N)$, $\lim_{q'_n \to 0} x_n(q_{-n}, q'_n; C) = x_n(q_{-n}, 0; C)$.*

Let $SI(N)$ be the set of scale invariant CSMs over $N \in \mathcal{N}$ in $CS(N,\infty)$. It is easy to characterize this set of scale invariant CSMs. Define $\Gamma_{SI}(N)$ to be the set of paths such that $\gamma(t;q) = \tau_q(\gamma(t;1))$, thus each component is $\gamma_i(t;q) = \gamma_i(t,1)q_i$, where 1 represents the unit vector $(1,1,\ldots,1)$.

[7] Note that while continuity at 0 is quite strong on its own, it is a relatively weak addition to ordinary dummy and additivity. See [9] for a discussion on the need for it and the changes that arise without it.

Theorem 2 *For any $N \in \mathcal{N}$ the following are equivalent:*

i) $x \in SI(N)$ and is continuous at 0.

ii) There exists a non-negative probability measure μ on $\Gamma_{SI}(N)$ such that

$$x = \int_{\gamma \in \Gamma_{SI}(N)} x^{\gamma} d\mu(\gamma).$$

Proof: This follows upon noting that a scale invariant CSM is completely determined by its behavior on any particular q, such as $q = 1$. Thus $SI(N)$ is isomorphic to $CS(1, N, \infty)$. $\square$

3 Consistent Cost Sharing Methods

In this section we provide sufficient conditions for a CSM to be consistent. We begin with a formal definition of consistency.[8] For cost sharing problems (as in cooperative games) there are many ways to define efficiency. Our definition is a very mild version of consistency that is based on the idea that if a single agent were to leave the problem taking her demand with her and paying her share, then the shares for the remaining agents should be unchanged. This is a natural extension of the version used by Hart and Mas-Colell [13] for TU games and by Moulin and Shenker [20] for cost sharing problems in which the cost function is required to be homogeneous. It is slightly weaker than the definition of consistency used by Mclean, Pazgal, and Sharkey [15] who interpret each unit of demand as an agent, and thus require (essentially) that fractions of an agent (in our sense) can depart, paying their share.

Nonetheless, this definition is still quite strong, and we will introduce a weaker condition (dummy consistency) below.[9]

[8] See, e.g. Thomson [26] for a survey of consistency and its definitions.

[9] For comparison, dummy consistency is slightly stronger than Mclean, Pazgal, and Sharkey's [15] definition of weak consistency.

Definition 6 (Consistency) *A CSM, $x \in CS$, is consistent if for any $N \in \mathcal{N}$, $q \in Q(N)$, $C \in \mathcal{C}(N)$, and $m, n \in N$, with $m \neq n$:*

$$x_n(q, C) = x_n(q_{-m}, R_m^{q_m}(C))$$

when $R_m^{q_m}(C)(p_{-m}) = C(p_{-m}, q_m) - x_m(p_{-m}, q_m; C)$ is an element of $CS(N \setminus m)$.[10]

Thus, any agent and her cost share, $x_m(p_{-m}, q_m; C)$, can be removed from the cost function without affecting the cost shares of the remaining agents, based on applying the same cost sharing mechanism to the reduced cost function, $R_m^{q_m}(C)(p_{-m})$. We will denote the subset of CS which is consistent by CON.

By contrast, we now introduce the property of dummy consistency which only requires that dummies can be removed from a cost sharing problem without affecting the cost shares, a much milder requirement.

Definition 7 (Dummy Consistency) *A CSM $x \in CS$ is dummy-consistent if for all $N \in \mathcal{N}$, $C \in CS(N)$ such that agent $n \in N$ is a dummy agent ($\partial_n C(p) \equiv 0$), $x_m(q; C) = x_m(q_{-n}, R_n^{q_n}(C))$ for all $m \neq n$.*

Note that when i is a dummy agent, then $R_i^{q_i}(C)(q_{-i}) = C(q_{-i}, 0_i)$, and thus we do not need to impose any monotonicity or differentiability conditions on $R_i^{q_i}(C)(q_{-i})$ in the definition. Let $DCON$ denote the subset of CS which is dummy consistent.

[10]This restriction is significantly less restrictive than it might appear. First, the requirement that $R_m^{q_m}(C)(p_{-m})$ be nondecreasing is irrelevant, since one can always add a term of the form $F = \sum_n f(x_n)$ to C to guarantee that $R_m^{q_m}(C + F)(p_{-m})$ is monotone. But then additivity and dummy render the new term irrelevant. In addition, the potential lack of smoothness is often unimportant, since one can take sequences of cost functions converging to the true cost function for which the "reduced" function is smooth, and the limiting process is well behaved. (See [8] for examples of this technique.)

Dummy consistency seems (to us) to be an extremely natural requirement for CSMs, while strong consistency, which may be desirable, is not be as fundamental.[11]

3.1 Consistent Cost Sharing Methods

In this section we develop sufficient conditions (based on Theorem 1) for consistency and dummy consistency. This will allow us to easily show that many well known CSMs are consistent, and to allow us to construct other consistent CSMs. We first introduce two new collection of paths.

Definition 8 *A path $\gamma \in \Gamma(N)$, for $N \in \mathcal{N}$ is separable if for each $n \in N$, $\gamma_n(t;q)$ is independent of q_{-n}, which we write as (with a slight abuse of notation) $\gamma_n(t;q_n)$.*

Definition 9 *A path $\gamma \in \Gamma(N)$, for $N \in \mathcal{N}$ is* associative *if it is separable and right differentiable in t, and there exists a continuous and right differentiable function $\phi(t,s)$ which is nondecreasing in both arguments such that, $\gamma_n(t;\gamma_n(s;q_n)) = \gamma_n(\phi(t,s);q_n)$ for all $n \in N$.*

Let $B \in \{A,S\}$, then let $\Gamma_B(N)$ be the subset of Γ containing all the separable paths when $B = S$ or associative paths, when $B = A$, and $\Gamma_B = \bigcup_{N\in\mathcal{N}} \Gamma_B(N)$. Note that there is a natural projection from $\Gamma_B(N)$ to $\Gamma_B(N \setminus m)$ for $m \in N$, π_m^N which simply removes the component γ_m from $\gamma \in \Gamma_B(N)$. This is well defined, since none of the components in $\Gamma_B(N \setminus m)$ depend on q_m.

[11] Several readers of earlier drafts of this paper have suggested that dummy consistency should actually be called "strong dummy" as it is essentially a version of the dummy axiom and much weaker than a consistency action. We retain the (perhaps misleading) title of dummy consistency to stress its relationship with ordinary consistency.

Theorem 3 *Let $B \in \{A, S\}$. Assume that for all $N \in \mathcal{N}$ there exists a probability measure μ on $\Gamma_B(N)$ such that for any $m \in N$,*

$$x^N = \int_{\gamma \in \Gamma_B(N)} x^\gamma d\mu(\gamma)$$

and

$$x^{N \backslash m} = \int_{\gamma \in \Gamma_B(N)} \pi_m^N[x^\gamma] d\mu(\gamma)$$

then $x \in DCON$ when $B = S$ and $x \in CON$ when $B = A$.

Proof: In appendix.

This result immediately implies that the previously discussed methods are consistent.[12]

Corollary 1 *The following CSMs are consistent (and thus dummy consistent):*

1) Aumann Shapley and weighted Aumann-Shapley.

2) Serial Cost and both linearly and exponentially weighted Serial cost.

3) Random order methods and Shapley-Shubik.

However, it is easy to show that not all CSMs are dummy consistent. Consider the case when $N = \{1, 2, 3\}$, $\overline{q} = 1$ and the CSM is x^γ where $\gamma(t; q) = (tq_1, tq_2, (1 - q_1)q_3 t + q_1 q_3 t)$, which is not separable. It is straightforward to show that this method is not dummy consistent.

4 Consistent Extensions

As mentioned in the introduction, one important question about consistency is whether two agent CSMs can be extended, using consistency, to general CSMs with an arbitrary number

[12]The boundedness of q in the above theorem does not affect these results, since we need only check at each finite value of q.

of agents. In [13], Hart and Mas-Collel, showed that in the case of binary demands, if we assume that a consistent CSM when restricted to two agents "divides the surplus equally" then it must be the Shapley Method. For the continuous demand setting considered in this paper, Mclean, Pazgal and Sharkey [15], showed that if a consistent CSM is the Aumann-Shapley method on all 2 agent problems, then it must be the Aumann-Shapley method. Both proofs require the use of a potential, and do not appear to be easily extendable for studying consistency questions related to extensions of other CSMs.

In this section, we show that when restricted to the case of additive CSMs, problems of this sort are straightforward to analyze by using the results in the previous section. In particular, we will often be able to use dummy consistency to simplify the analysis. Furthermore, the results for dummy consistency also apply to the case with a fixed population where we might have conditions on the CSM when there are a group of dummy agents. For example, consider a three agent cost sharing problem for which we know that when agent one agent is a dummy, the other 2 agents should use the Aumann-Shapley method, but when a different agent is a dummy, then the remaining two should use serial cost. Using the tools developed in this section for dummy consistency, we can show that there is a unique CSM which satisfies these conditions.

Thus, our goal in this section is to understand when a CSM defined for two agent problems extends to a consistent CSM for an arbitrary number of agents. Let $\mathcal{N}_2 = \{N \in \mathcal{N} \mid |N| = 2\}$ and define $CS_2 = \bigcup_{N \in \mathcal{N}_2} CS(N)$, which is the set of CSMs defined for all two agent methods.

The basic problem that we now consider is this: given an $y \in CS_2$, what is the set of consistent extension of y, i.e., what methods $x \in CON$ (or $DCON$) coincide with y on all

$N \in \mathcal{N}_2$.

Note that we are allowing asymmetry in the methods used for the various two player coalitions.[13] For example, one pair might be using the Shapley-Shubik method while a different pair might be using the Serial method.

From the previous section we know that if y cannot be generated by associative (resp. separable) paths then typically there can be no consistent (resp. dummy consistent) extensions of y to CS. However, even when y is generated in this way, there may be no extensions or many extensions. We first show that in a common setting there is at most one consistent extension. (We give examples of methods with multiple extensions in the following section.)

Theorem 4 *Let $x \in CS_2$ be generated by a single associative (resp. separable) path. Then there exists at most one $\hat{x} \in CS$ such that $x = \hat{x}$ on $\mathcal{N}_2$ and is consistent (resp. dummy consistent).*

Proof: Let $\hat{x}$ be one such extension. Then x is generated by a sum of separable paths γ^k; however, if the sum cannot be written as a single path, then there must be some $N \in \mathcal{N}_2$ for which the associated CSM is not constructed from a single path, violating the assumption. Thus, all extensions must be path methods, but clearly if there were two paths that differed, then they must differ on some $N \in N_2$. $\square$

This theorem is applicable to almost all of the previously discussed methods.

Corollary 2 *The following methods defined on CS_2 extend uniquely to $DCON$ (resp. CON): Incremental methods, Aumann-Shapley, Serial cost, and their weighted versions.*

[13] See [19] for a discussion of related asymmetric methods in rationing problems.

Note that the above theorem does not include Shapley-Shubik, since it is a sum of multiple path generated methods. As we show in the next section Shapley-Shubik has many (nonsymmetric) extensions; however, it has only one symmetric extension.

Theorem 5 *The Shapley-Shubik CSM defined on CS_2 has a unique consistent (resp. dummy consistent)* symmetric *extension.*

Proof: Fix $N = \{1, 2, \ldots, n\} \subset Z^+$ such that $n \geq 3$. For simplicity assume that $t \in [0, n]$. For any $i, j \in N$ with $i \neq j$, the Shapley-Shubik CSM on $\{i, j\}$, $CS(\{i, j\})$ can be written as $\sum_{\sigma \in \Sigma} x^{\gamma^\sigma}/n!$, where for $n \in N$, $\gamma_n^\sigma(t; q) = \phi^{\sigma(i)}(t) q_{\sigma(i)}$. It is easy to see that when viewed as a CSM over N, these are the only paths (up to equivalence of reparameterization) which "project" for all $i, j \in N$, $i \neq j$, to a two agent random order value. Thus any extension to N can be written as $\sum_{\sigma \in \Sigma} \lambda(\sigma) x^{\gamma^\sigma}$ where $\sum_{\sigma \in \Sigma} \lambda(\sigma) = 1$ and for all $\sigma \in \Sigma$, $\lambda(\sigma) \geq 0$. However, unless all the λ's are equal the method is not be symmetric. $\square$

4.1 Nonsymmetric extensions

Somewhat surprisingly, a nonsymmetric CSM may be symmetric when "projected" onto N_2. For example, as we now demonstrate, the Shapley-Shubik mechanism has (many) nonsymmetric extensions.

Consider the case when $N = \{1, 2, 3\}$. We construct a CSM which is a convex combination of Incremental methods with the following orderings: $\omega^1 = (1, 2, 3)$ and $\omega^2 = (3, 2, 1)$. Define x^i to be the random order value for order σ^i. Then consider the CSM defined by $(x^1 + x^2)/2$. It is straightforward to check that all two agent methods induced by this method coincide with the Shapley-Shubik CSM. Thus, this provides an example of a nonsymmetric extension of the Shapley-Shubik CSM. (Agents 1 and 3 are interchangeable, but agent 2 is not with either 1 or 3.) Note that there are other nonsymmetric extension which are combinations of various random order values.[14]

In general there are even more complex examples of CSMs with non-unique consistent extensions. Note that generically, if $x \in CS_2$ is generated by a finite number of paths then the extension (if it exists) will be unique, since the paths can not be 'mixed' together; however, in particular cases they can have multiple extensions as shown here. Consider the following asymmetric CSM in CS_2 which is generated by the average of the following associative and scale invariant paths:

$$\gamma_a(t;q) = (tq_1, tq_2, t^2 q_3, tq_4, tq_5, \ldots),$$

$$\gamma_b(t;q) = (tq_1, t^{1.5} q_2, t^6 q_3, tq_4, tq_5, \ldots),$$

$$\gamma_c(t;q) = (tq_1, t^{0.5} q_2, t^3 q_3, tq_4, tq_5, \ldots),$$

where for simplicity we require that $t \in [0,1]$. By this we mean that for $N = \{1,2\}$ the CSM is the average of the methods generated by the three paths, (tq_1, tq_2), $(tq_1, t^{1.5}q_2)$, and $(tq_1, t^{0.5}q_2)$.

Clearly, this element of CS_2 has the "natural" extension $[x_{\gamma_a} + x_{\gamma_b} + x_{\gamma_c}]/3$. However, it also has another extension which is given by $[x_{\gamma_d} + x_{\gamma_e} + x_{\gamma_f}]/3$, where

$$\gamma_d(t;q) = (tq_1, tq_2, t^6 q_3, tq_4, tq_5, \ldots),$$

$$\gamma_e(t;q) = (tq_1, t^{1.5} q_2, t^3 q_3, tq_4, tq_5, \ldots),$$

[14] We only considered a world with 3 agents in this example to simplify notation. This can be easily "embedded" in the world with an infinite number of agents which we have been considering using the same technique as demonstrated in the next example.

$$\gamma_f(t;q) = (tq_1, t^{0.5}q_2, t^2q_3, tq_4, tq_5, \ldots);$$

however they both have the same projections onto 2 agent methods.

Finally, any convex combination of these two methods is dummy consistent. Thus, there are infinitely many "nontrivial" extensions. In general, extensions of arbitrary CSMs can be quite complex.

A Appendix: Proofs of theorems

A.1 Theorem 1

Theorem 1: For any $N \in \mathcal{N}$ and $q \in Q(N)$, the following are equivalent:

i) $x \in CS(q, N)$.

ii) There exists a probability measure μ on $\Gamma(q, N)$ such that

$$x = \int_{\gamma(\cdot;q)\in\Gamma(q,N)} x_{\gamma(\cdot;q)} d\mu(\gamma(\cdot;q)).$$

Proof: Fix $N \in \mathcal{N}$ and $q < \overline{q}$. First we discuss some mathematical preliminaries. Let $x, y \in CS(q, N)$ and let ρ^x, ρ^y be their representing vector measures, as in Proposition 1. Then define $\hat{d}(x, y) = \sum_{n\in N} d(\rho^x_n, \rho^y_n)$, where $d(\cdot, \cdot)$ is the Prohorov metric [3]. Under this metric, convergence for $CS(q, N)$ is given by $\lim_{t\to\infty} x^t = x$ iff for all $C \in \mathcal{C}(N)$, $\lim_{t\to\infty} x^t(q; C) = x(q; C)$. Note that this is the metric of weak convergence and thus $CS(q, N)$ is a closed and compact metric space, as it is isomorphic to a product of closed convex subsets of the space of probability measures. (See [22] for the standard construction.)

Using this, we define the following metric on $\Gamma(q, N)$: if $\gamma, \gamma' \in \Gamma(q, N)$ then $\overline{d}(\gamma, \gamma') = \hat{d}(x^\gamma, x^{\gamma'})$. In this metric, $\Gamma(q, N)$ is a compact metric space. Note also that in these metrics the mapping that maps γ to x^γ is injective and continuous as is the mapping which maps x to its related measure ρ.

Now we construct the representing measure in the theorem: Let $CS(q, N; k)$ be the set of $x \in CS(q, N)$ for which their representing measures ρ have support only on the grid $g_k = \{p \mid \exists n \in N, \;\; s.t. \; \forall m \neq n, \; 2^k p_m \in Z\}$ (this is a grid and not a lattice). Note that $g_k \subset g_{k+1}$ and therefore $CS(q, N; k) \subset CS(q, N; 2k)$ for all $k \geq 1$. Similarly define $\Gamma(q, N; k)$ which is the set of paths which have support on g_k, for which $\Gamma(q, N; k) \subset \Gamma(q, N; 2k2)$ for all $k \geq 1$. It is straightforward to show that $CS(q, N; k)$ is the set of CSMs which only depend on the value of the cost function at the intersection points of the grid, i.e., $\hat{g}_k = \{p \mid \forall n \in N, \;\; 2^k p_n \in Z\}$. This is the problem considered by Wang [27]. In that paper he proves the following using an elegant argument based on a result of Sprumont [25]:

Proposition 2 (Wang (1999)) *The set ext $CS(q, N; k)$ is equal to the set $\{x^\gamma \mid \gamma \in \Gamma(q, N; k)\}$.*

Note that $\Gamma(q, N; k)$ is finite; thus for each $x^k \in CS(q, N; k)$ there exists a vector probability measure μ^k on $\Gamma(q, N; k)$ such that $x^k = \int_{\gamma \in \Gamma(q,N;k)} x^\gamma d\mu^k(\gamma)$. Now, by taking the "limit" as $k \to \infty$ we get the desired result, after noting the following.

Lemma 1 *Given an $x \in CS(q, N)$ there exists a sequence $x^k \in CS(q, N; k)$ such that $\lim_{k\to\infty} x^k = x$.*

Proof: Let $\mathcal{C}^\dagger(q, N)$ be the subset of $\mathcal{C}(q, N)$ which contains only functions for which all cross partial derivatives are continuous, i.e., the functions $\partial_S C(q)$ is continuous for all $S \subseteq N$.

Note that $\mathcal{C}^\dagger(q,N)$ is dense in $\mathcal{C}(q,N)$ and also that any function in $\mathcal{C}^\dagger(q,N)$ can be uniquely decomposed into a sum of funtions $C = \sum_{S\subseteq N} C^S$, where C^S has support only on the interior of $[0,q_S]$.

Following [9] (Lemma 1) starting with the representation formula:

$$x_n(q;C) = \int_{[0,q]} \partial_n C(p) d\rho_n(q)(p)$$

we can integrate by parts multiple times to get

$$x_n(q;C) = \sum_{S\subseteq N} (-1)^{(1+|S|)} \int_{[0,q_S]} \partial_S C^S(p) \lambda_n^S(p) dp_S$$

where $\partial_S C^S(p)$ is $(\Pi_{m\in S}\partial_m)C^S(p)$ and λ_n^S is an L^1 density function on $[0,q_S]$ satisfying $\sum_{m\in S}\lambda_m^S(p) = 1$. (Note that this is the continuous version of the discrete representation formula in [18] discussed in [9].)

Now, given such a set of λ's which represent the CSM x, construct a CSM, x^k, as follows. Let $H^S(p) \subset [0,q_S]$ be the hypercube in the grid g_k which contains $p \in [0,q_S]$, i.e., let p_- be the largest p in $\hat{g}_k$ which is not larger (in the vector ordering) than p and p_+ to be the smallest element of $\hat{g}_k$ which is not smaller than p. Now define $\lambda_m^S(p;k) = \int_{[p_-,p_+]} \lambda_m^S(p)/(p_+ - p_-)^{|S|}$ and note that by construction the CSM induced by these $\lambda_m^S(p;k)$'s only depends on the values of $C(p)$ on $\hat{g}_k$ and thus defines an element of $CS(q,N;k)$. Also, note that since it only depends on the values of $C(p)$ on $\hat{g}_k$, it is well defined on all of $\mathcal{C}(q,N)$ and not just on $\mathcal{C}^\infty(q,N)$. Also, on all $C \in \mathcal{C}^\dagger(q,N)$ $\lim_{k\to\infty} x^k(q;C) = x(q;C)$. Since $\mathcal{C}^\dagger$ is dense in $\mathcal{C}$ this implies that $\lim_{k\to\infty} d(x^k,x) = 0$. $\diamond$

Proof of theorem: Consider some $x \in CS(q,N)$ and choose any sequence $x^k \in CS(q,N;k)$ such that $\lim_{k\to\infty} x^k = x$. Now, using the above proposition, there exists a sequence of vector probability measures μ^k which represent the CSMs x^k and note that μ^k induces a measure on $\Gamma(q,N)$ which we will also (with a slight abuse of notation) denote by μ^k. Since $\Gamma(q,N)$ is a compact metric space, the space of probability measures on $\Gamma(q,N)$ must be compact [22], and therefore there exists a subsequence on which μ^k converges to a measure on $\Gamma(q,N)$. This measure is the desired representation of x as an integral over $\Gamma(q,N)$, proving the theorem. □

B Theorem 3

Let $B \in \{A, S\}$. Assume that all $N \in \mathcal{N}$ there exists a probability measure μ on $\Gamma_B(N)$ such that for any $m \in N$,

$$x^N = \int_{\gamma\in\Gamma_B(N)} x^\gamma d\mu(\gamma)$$

and

$$x^{N\backslash m} = \int_{\gamma\in\Gamma_B(N)} \pi_m^N[x^\gamma] d\mu(\gamma)$$

then $x \in DCON$ when $B = S$ and $x \in CON$ when $B = A$.

Proof: It is easy to see that a CSM generated by a separable path is dummy consistent and thus a convex combination of such methods is also dummy consistent. We now show that the CSM generated by an associative path is consistent, completing the proof.

Lemma 2 *If γ is associative then $\gamma_n(t, q_n)$ is nondecreasing and right differentiable in q_n. In addition*

$$\partial_s^+\phi(t,s) = \partial_n^+\gamma_n(t;\gamma_n(s;q_n))\dot{\gamma}_n(s;q_n)/\dot{\gamma}_n(t;\gamma_n(s;q_n)),$$

where ∂^+ denotes right differentiation and $\dot{\gamma}_n(t;q_n) = \partial_t^+\gamma_n(t;q_n)$, Proof: Given q_n choose any $q_n' > q_n$ and let s solve $\gamma(s, q_n') = q_n$ and s solve $\gamma_n(s, q_n') = q_n'$ and note that $s' > s$. Then by associativity $\gamma_n(t; q_n) = \gamma_n(t, \gamma_n(s; q_n') = \gamma_n(\phi(t, s); q_n')$ and similarly $\gamma_n(t, q_n') = \gamma_n(\phi(t, s'), q_n')$, but since $\phi(t, s') \geq \phi(t, s)$ which implies that $\gamma_n(t; q_n') \geq \gamma_n(t; q_n)$ since $\gamma_n(\cdot, \cdot)$ is nondecreasing in its first argument by definition.

Differentiating the identity $\gamma_n(t; \gamma_n(s; q_n)) = \gamma_n(\phi(t, s); q_n)$ with respect to s yields

$$\partial_n \gamma_n(t; \gamma_n(s; q_n)) \dot{\gamma}_n(s; q_n) = \dot{\gamma}_n(t; \gamma_j(s; q_j)) \partial_s \phi(t, s),$$

thus since all the other right derivatives exist, the right derivative of $\gamma_n(t, q_n)$ with respect to q_n must exist and solve the equation. $\diamond$

Lemma 3 *If γ is an associative path, then x^γ is strongly consistent.*

Proof: Note that since all the functions involved (γ and ϕ) are nondecreasing and right differentiable, all the functional compositions which we use in the following calculations are also nondecreasing and right differentiable. Thus, in the following we use right differentiation, denoted by ∂^+. Also, to simplify notation assume that $\gamma(1, q) = q$.

Assume that C is twice continuously differentiable and that $n \neq m \in N$. Now

$$x_n(q; C) = \int_0^1 dt\ \partial_n^+ C(\gamma(t; q))\ \dot{\gamma}_n(t; q_n),$$

where we are using the separability of γ. Differentiating yields,

$$\partial_m x_n(q; C) = \int_0^1 dt\ \partial_{nm} C(\gamma(t; q))\ \dot{\gamma}_n(t; q_n)\ \partial_m^+ \gamma_m(t; q_m). \qquad (*)$$

Strong consistency requires that $x_m(q; C) = x_m(q_{-n}; \hat{C}) - x_m(q_{-n}; C')$, where $C'(p_{-n}) = x_n(p_{-n}, q_n; C))$, $\hat{C}(p_{-n}) = C(p_{-n}, q_n)$, and we ignore the fact that C' is not nondecreasing, as it does not affect the calculations. Now we compute the second term. Using (*) yields

$x_m(q_{-m}; C') =$

$$\int_0^1 ds \int_0^1 dt\ \partial_{nm} C(\gamma_{-n}(t; \gamma_{-n}(s; q_{-n})), \gamma_n(t; q_n))\ \dot{\gamma}_n(t; q_n)\ \dot{\gamma}_m(t; q_m)\ \partial_m^+ \gamma_m(t; \gamma_m(s; q_m))$$

and since γ is associative

$$x_m(q_{-m}; C') = \int_0^1 ds \int_0^1 dt\ \partial_{nm} C(\gamma_{-n}(\phi(t,s); q_{-n})), \gamma_n(t; q_n))\ \dot{\gamma}_n(t; q_n)\ \dot{\gamma}_m(t; q_m)\ \partial_m^+ \gamma_m(t; \gamma_m(s; q_m))$$

Changing variables to $u = \phi(t, s)$ and $v = t$, noticing that the Jacobian of the transformation is $\partial_s^+ \phi(t, s)$, and using the identities, $\phi(t, 0) = 0$ and $\phi(t, 1) = t$ results in

$$x_m(q_{-n}; C') = \frac{\int_0^1 du \int_u^1 dv\ \partial_{nm} C(\gamma_{-n}(u; q_{-n}), \gamma_n(v; q_n))\ \dot{\gamma}_n(v; q_n)\ \dot{\gamma}_m(s; q_m)\ \partial_m^+ \gamma_m(u; q_m)}{\partial_s^+ \phi(t, s)} (**).$$

Combining the formula from the preceding lemma (with m replacing n and v replacing t) with (**) yields

$$x_m(q_{-n}; C')) = \int_0^1 du\ \int_u^1 dv\ \partial_{nm} C(\gamma_{-n}(u; q_{-n}), \gamma_n(v; q_n))\ \dot{\gamma}_n(v; q_n)\ \dot{\gamma}_m(u; q_m).$$

Integrating with respect to v yields

$$x_m(q_{-n}; C') = \int_0^1 du \partial_m [C(\gamma_{-n}(u; q_{-n}), q_n) - C(\gamma(u; q))] \dot{\gamma}_m(u; q_m)),$$

which equals $x_m(q; C) - x_m(q_{-n}; \hat{C})$, proving strong consistency for twice differentiable C. Since the set of twice differentiable functions is dense in $\mathcal{C}(N)$ this suffices to prove the lemma. □

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