Dynamic Monotonicity and Comparative Statics for Real Options*

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Abstract

We analyze the relationship between complementarities and monotonicity in dynamic programs. While complementarities (supermodularity and nondecreasing differences) are sufficient for monotonic statics in stochastic dynamic programs, they are only 'robustly' necessary in a limited way under heavy discounting. Monotonicity of the decision rule does *not* in general imply monotonicity of the optimal path. In fact, optimal paths in such models may exhibit extremely complex behavior. For convex adjustment costs many real options models have monotone decision rules, while for fixed costs they typically do not. We also strengthen a well known characterization of complementarities due to Milgrom and Shannon [19]. *Journal of Economic Literature* Classification Numbers: C61; D92; E22.

1 Introduction

In this paper we study the comparative statics and monotonicity properties of solutions to dynamic programming problems. We attempt to understand the conditions under which the optimal paths and comparative statics are well behaved. In particular, we are interested in models which arise in the study of real options.[1]

In general, dynamic programming problems can give rise to extremely complicated solutions in which it is not possible to make clear statements about the response of the optimal solution to changes in exogenous parameters or to describe the optimal path simply.[2] This leads to many fundamental problems in interpretation. For example, Denekere and Pelikan [5] argue that in such cases optimal behavior would not be observed in practice.

We begin with an analysis of the comparative statics for optimal decision rules.[3] We show that complementarities (supermodularity and nondecreasing differences) are sufficient to guarantee monotone statics for the decision rule. It is possible for models without complementarities to have monotone decision rules; however, unless there is some minimal level of complementarities (to be defined formally in Section 3) these results will not be robust.

[1] Dixit and Pindyck [8] provides a broad survey of such models.

[2] Boldrin and Montrucchio [3] show that there exist dynamic programs which display the behavior of any (one dimensional) nonlinear mapping, and thus can be chaotic and depend sensitively on both initial conditions and exogenous parameters.

[3] Milgrom and Shannon [19] have analyzed the static deterministic case, and Athey [2] has extended their analysis to the stochastic case.

(We define robustness as in Milgrom [16] and Milgrom and Roberts [18] who imbed a single decision problem in an entire family of such problems to reflect the modeler's uncertainty about specific aspects of the problem). The necessity of (limited) complementarities for monotone statics results is actually dependent on heavy discounting. For low discount rates, problems without complementarities may give rise to robust monotone decision rules.

Intuitively, one might expect that monotonicity of the decision rules would imply monotonicity (or at least simplicity) of the optimal path. While this statement is true in one dimension, it requires a significant qualification in higher dimensions, since optimal paths arising from monotone decision rules can be extremely complicated. We give an example in two dimensions in which an optimal path for a monotone decision rule is chaotic. The existence of chaotic paths in such models is structurally stable (i.e. slight perturbations of the model will also have optimal paths which are chaotic).

However, while these models have optimal paths which are chaotic for some initial condition, these paths appear to be rare in the sense that for most initial conditions the paths are not chaotic. This is shown by example and is based on a conjecture by Friedman and Landsberg [12]. If this conjecture is correct then even though chaos occurs in many of these models, it may be irrelevant.

In the next section we give a simple motivating example. Section 3 analyzes the necessary and sufficient conditions for monotonicity while Section 4 studies the structure of optimal paths of monotone decision rules. Section 5 describes the connection between the general theory and real options.

2 Example: Optimal Stopping

The decision-maker in the simplest optimal stopping problem makes a single discrete choice. For concreteness, we will discuss a firm's decision to scrap an existing project. The standard model is as follows. In each period the firm can exercise the option to scrap and receive a "termination payoff" which is Ω. Alternatively, the firm can continue to wait and receive $\pi(x^t)$ per period. We assume that $x^t \in \Re$ is a random variable given by a stationary Markov process, and has positive persistence of uncertainty, i.e. larger x^{t-1}'s lead to larger x^t's, in the sense of first order stochastic dominance. Note that persistence is satisfied by many stochastic processes commonly studied, such as Brownian motion with drift and even many mean reverting processes. Let the discount factor be δ.

Now we ask when the optimal stopping rule is nondecreasing, which corresponds in this case to the existence of a threshold x^*, such that the first time that $x^t \leq x^*$ the firm will scrap the project. (Non-increasing stopping rules can be handled analogously.)

Theorem 1 *If $\pi(x^t)$ is nondecreasing in x^t, then the stopping rule is nondecreasing.*

Proof: See for example [8]. $\Box$

We now cast this problem in a form amenable to analysis by lattice techniques. We define the strategy space of the underlying optimization problem to be $S = \{0, 1\}$, where 1 denotes continuation and 0 denotes scrapping. The firm's payoff function per period, $\hat{\pi}(s^t, x^t)$ is $\pi(x^t)$ if $s^t = 1$ and $\hat{\Omega}$ if $s^t = 0$, where we choose $\hat{\Omega}$ such that its net present value is Ω, i.e. $\hat{\Omega} = (1 - \delta)\Omega$.

In this example, once a firm has chosen to scrap ($s^t = 0$), it cannot restart ($s^t = 1$). We model this by assuming that the cost to restarting is infinite. This is not crucial to our analysis; in fact we can allow the cost of restarting to be any non-negative number.

Now we can restate a general version of the previous result:

Definition 1 *$\hat{\pi}(s^t, x^t)$ has nondecreasing differences in x^t if $\hat{\pi}(1, x^t) - \hat{\pi}(0, x^t)$ is nondecreasing in x^t.*

Note that this is identical to the assumption in Theorem 1, since $\hat{\pi}(0, x^t) = \hat{\Omega}$.

Theorem 2 *Assume that $\hat{\pi}(s^t, x^t)$ has nondecreasing differences in x^t. Then the stopping rule is nondecreasing in x^t.*

Proof: This follows directly from Theorem 4 stated below. □

Notice that Theorem 1 is an immediate consequence of this result. However, nondecreasing differences in $\hat{\pi}(s^t, x^t)$ (equivalently, monotonicity of $\pi(x)$) is not necessary for a nondecreasing stopping rule, but is necessary in for 'robust' conclusions. We demonstrate this by example.

Assume that there is a fixed per period cost k of continuing a project, such as overhead. The next theorem shows that the only way the stopping rule can be nondecreasing, for any k, is to have nondecreasing differences in x.

Theorem 3 *The following statements are equivalent:*

(1) $\hat{\pi}(s^t, x^t)$ *has nondecreasing differences in* x^t.

(2) For all $\delta > 0, k \in \Re$, *the decision rule is monotone.*

Proof: This follows directly from combining Theorem 4 and Theorem 5, stated below. □

Thus we have determined that nondecreasing differences are both necessary and sufficient for the stopping rule to be monotone. They are sufficient in the sense that if there are unknown parameters of the problem, such as fixed per period costs, then for any problem which does not have nondecreasing differences there exists a cost such that the stopping rule is not monotone. In the next section we will show the generalization of this result to more complicated dynamic decision problems.

3 Monotonicity of Decision Rules

Consider a general dynamic decision problem. Let S be the set of actions. For simplicity we will assume that $S = S_1 \times \cdots \times S_n$ where $S_i \subseteq \Re$, and S_i is locally compact.[4] S is a lattice under the standard vector ordering; for $s, s' \in S$ $s \leq s'$ if $s_i \leq s'_i$ for all i. Note that the lattice is complete – for any closed and bounded set A of actions there exists a greatest lower bound (resp. least upper bound), written $\min(A)$ (resp. $\max(A)$). Given two actions s, s' the greatest lower bound is written $s \wedge s'$ and the least upper bound $s \vee s'$. Note that $s \wedge s'$ is the component wise minimum of s and s' (similarly for $s \vee s'$). Both the unit cube $[0,1]^n$ and the set $\{0,1\} \times \{0,1,2\}$ are examples of lattices under the standard vector ordering. The first can be interpreted as a finite set of projects, each of which can be unutilized ($s_i = 0$), fully utilized ($s_i = 1$), or partially utilized ($0 < s_i < 1$). The second can be interpreted as two projects with respectively 2 and 3 levels of action. Thus, we analyze both continuous and discrete action spaces with the same techniques.

We assume there is a finite set of random processes $x_i^t \in X_i$, where $X_i \subseteq \Re$ is locally compact and letting $X = X_1 \times \cdots \times X_m$ be the ordered set defined by the vector ordering. (X can be interpreted as the set of exogenous prices or market conditions.) These are assumed to be stationary, Markovian, and have positive persistence of uncertainty.

[4] A set S_i is locally compact if for any $s_i \in S_i$ there is an open set A containing s_i whose closure is compact. (See, e.g. [20].)

Assumption 1

i) Markovian: $(x_i^t | x^{t-1}, x^{t-2}...) = (x_i^t | x^{t-1})$.

ii) Persistence: if $x^{t-1} > y^{t-1}$*, for all* j*, then* $(x_i^t | x^{t-1})$ *is equal or larger (in the sense of first order stochastic dominance) than* $(x_i^t | y^{t-1})$.

Let the set of exogenous parameters be denoted by $\Theta = \Theta_1 \times \cdots \times \Theta_l$ where $\Theta_i \subseteq \Re$ is locally compact. We assume the vector ordering on both X and Θ.[5]

In each period, the set of allowable actions may depend on the action from the previous period. We denote this set by $S(s', x; \theta)$ and require the following.

Assumption 2

i) The graph $\Gamma = \{(s, s', x; \theta) \mid s \in S(s', x; \theta)\}$ *is a sublattice of* $S \times S \times X \times \Theta$ *(in the product order).*

ii) $S(s', x; \theta)$ *is a complete sublattice of* S.[6]

Note that this assumption implies that the sets $S(s', x; \theta)$ are nondecreasing (in s', x, θ), in the strong set order [23], defined below. For future reference, we define $\Gamma(x; \theta) = \{(s, s') \in S \times S \mid s \in S(s', x; \theta)\}$.

[5] More generally, both X and Θ could be any partially ordered set. However, this would complicate the notation unnecessarily.

[6] A set A is a sublattice of S if for all $a, b \in A$ both $a \wedge b$ and $a \vee b$ (taken in S) are elements of A.

Definition 2 *Let A, B be sublattices of S. We write $A \geq B$ if for all $a \in A, b \in B$ we have, $a \wedge b \in B$ and $a \vee b \in A$. Note that this implies that* $\min(B) \leq \min(A)$ *and* $\max(B) \leq \max(A)$.

In each period, the payoff is $F(s^t, s^{t-1}, x^t; \theta) + \sum_i \kappa_i(s_i^t)$, where $\kappa_i(s_i^t)$ can be interpreted as the per period operating cost of action s_i^t and will represent the notion of robustness for deriving necessity results.[7] (Both $F(s^t, s^{t-1}, x^t; \theta)$ and $\kappa_i(s_i^t)$ are assumed to be order upper semi-continuous throughout; see e.g. [17].)

A decision rule is a mapping from $S \times X \times \Theta$ to S, $\hat{s}(s^{t-1}, x^t; \theta)$. Given an initial condition s^0, x^0 the value of a decision rule is given by:

$$E\{\sum_{t=1}^{\infty} \delta^{t-1}[F(\hat{s}, s^{t-1}, x^t; \theta) + \sum_i \kappa_i(s_i^t)]\}$$

where we have suppressed the arguments of $\hat{s}$. We denote the optimal decision rule by $\hat{s}^*(s^{t-1}, x^t; \theta)$. However, since we have made no assumptions of convexity on either the payoff functions or the action space it is not necessarily true that the optimal decision is unique, and thus $\hat{s}^*(s^{t-1}, x^t; \theta)$ is a correspondence.

We will assume that the value function exists and is well defined. This is true in most applications, and under fairly weak assumptions, since persistent stochastic processes are typically well behaved. (See, e.g. [15, 21].)

[7] Note that we do *not* make the usual assumptions on F, typically concavity and monotonicity, as found in the standard texts, e.g. [21].

Definition 3 *A dynamic decision problem (DDP) is nondecreasing if* $\hat{s}^*(s^{t-1}, x^t; \theta)$ *is nondecreasing (in the strong set order) in* (s^{t-1}, x^t, θ).

Now we describe the conditions under which a dynamic decision problem is nondecreasing. (These definitions follow Topkis [22], and Milgrom and Roberts [17].)

Definition 4 *Let* B *be a lattice. A function* $f(b)$ *is supermodular if for all* $b, b' \in B$*:* $f(b) + f(b') \leq f(b \wedge b') + f(b \vee b')$.

Definition 5 *Let* B *be a lattice and* X *be a partially ordered set. A function* $g(b, x)$ *has nondecreasing (resp. increasing) differences in* B, X *with respect to* $S(\cdot)$ *if for all* $b \in B$ *and* $x > x' \in X$*,* $g(b, x) - g(b, x')$ *is nondecreasing (resp. strictly increasing) for all* $b \in S(x) \bigcap S(x')$.

Consider the case when $S(s', x; \theta) = s'$. Then for any F the decision rule must be monotone. Thus, we do not require any restrictions on F in regions that are irrelevant. This motivates the following definition.

Definition 6 *A DDP has "full complementarities" (resp. full strict complementarities) if* $F(s, s', x; \theta)$ *is supermodular in* s, s' *on* $\Gamma(x; \theta)$ *and has nondecreasing (resp. increasing) differences in* $((s, s'), (x, \theta))$ *with respect to* $S(s', x; \theta)$.

Note that when x^t are i.i.d. we can remove the assumption of nondecreasing differences in x. This is used in [15] to analyze a growth model. However, in general it cannot be dropped.

Theorem 4 *A DDP (satisfying Assumptions 1 and 2) is nondecreasing if it has full complementarities.*

Proof: This follows easily from [1, 15]. Thus we only provide a sketch of the proof.

The optimization can be recast in the form of a dynamic program. Let $V(s^{t-1}, x^t; \theta)$ be the expected value of the optimal decision rule from time t on given s^{t-1}, x^t, θ. Then $V(s', x; \theta) = TV(s', x; \theta)$ where

$$TV(s', x; \theta) = \max_{s \in S(s', x; \theta)} \left\{ F(s, s', x; \theta) + \sum_i \kappa_i(s_i) + \delta E\left(V(s, x'; \theta)|x\right) \right\} \quad (*).$$

Existence follows from standard contraction mapping arguments on T. Full complementarities follows since T preserves full complementarities.

The optimal decision rule is given by the maximum in (*) which is a supermodular optimization problem, and thus the solution is monotone in s^{t-1}, x^t, θ. (Note that measurability arguments are straightforward due to supermodularity and persistence; see [15], for details.) □

In order to show necessity we imbed a single problem in a family of problems which represent our uncertainty about the details of the model. We do not discuss this approach in detail here but instead refer the reader to Milgrom and Roberts [18]. However, we will demonstrate this approach via several examples in Section 5. Recall that in Section 2 the model uncertainty concerned the per period operating costs.

Below, we prove a general necessity result for when a DDP is nondecreasing. However, full complementarities are *not* necessary.

Definition 7 *A DDP has "limited complementarities" (resp. limited strict complementarities) if $F(s, s', x; \theta)$ is supermodular in s, s' on $\Gamma(x; \theta)$ and has nondecreasing (resp. increasing) differences in $(s, (x, \theta))$ with respect to $S(x; \theta)$.*

Theorem 5 *If for all $\delta > 0$ and all order upper semi-continuous functions $\kappa_i(s_i)$ a DDP is nondecreasing then it has limited complementarities.*

Proof: We will first show necessity for $\delta = 0$ and then extend this to $\delta > 0$ by continuity.

Consider the case when $\delta = 0$. In this case the solution is nondecreasing for any function κ_i if and only if the function being optimized has limited complementarities. This is a slight strengthening of a result in Milgrom and Shannon [19], and is given in the appendix. Thus, we can find a set of $\kappa_i(s_i)$ and $(s, x, \theta) > (s', x', \theta')$ for which $\max(\hat{s}^*(s, x; \theta)) < \min(\hat{s}^*(s', x'; \theta'))$ Therefore, for δ sufficiently small this ordering is preserved.[8] $\square$

[8]This can be most easily seen by direct examination of the mapping T defined in the previous proof.

3.1 Low Discounting

Note that the previous result requires that the decision rule be monotone for all possible discount rates, and in particular the proof only shows that necessity occurs for heavy discounting. The following example shows that the above necessity result is false when there is low discounting (i.e. δ close to one).

Consider a deterministic version of the optimal stopping problem with $(x^t|x^{t-1}) = x^{t-1}+1$, and for simplicity assume that x^0 is an integer. Now let $\pi(1,x) = x$ for $x \geq 0$, $\pi(1,x) = x+2$ for $x < 0$ and $\pi(0,x) = 0$, and assume the cost of restarting is infinite. (Recall that $s = 1$ corresponds to continuation and $s = 0$ is scrapping.)

It is easy to see that for δ close to 0 and $k = -1/2$ the optimal decision rule $\hat{s}^*(s^{t-1}, x^t)$ is not monotone since $\hat{s}^*(1,-1) = 1$, $\hat{s}^*(1,0) = 0$, and $\hat{s}^*(1,1) = 1$. (Note that $\hat{s}^*(0,x) = 0$ since restarting is not allowed.) However, for low discounting this problem is monotone.

For $x \geq 0$, $\pi(s^t, x^t)$ has increasing differences, and therefore the stopping rule will be monotone. A straightforward computation yields: $V(1,0) = \max[0, \delta^2/(1-\delta)^2 - k]$, and it is optimal to stop only if $k \geq \delta^2/(1-\delta)^2$.

Now $V(1,-1) = \max[0, 1+\delta V(1,0) - k]$ and it is optimal to stop only if $k \geq 1 + \delta V(1,0)$. In order for the stopping rule not to be monotone, k must satisfy: $k < \delta^2/(1-\delta)^2$ and $k > 1 + \delta^3/(1-\delta)^2$. Thus for $\delta > (\sqrt{(5)} - 1)/2 \approx 0.618$ (the golden ratio)the stopping rule is monotone.[9] Intuitively, the long term payoffs for higher values of x are sufficient to overwhelm the current loss incurred when $x = 0$.

Thus, for δ close to 1 we get a monotone decision rule for a payoff function $\pi(s,x)$ which does not have nondecreasing differences.

4 The Structure of Optimal Paths

In the previous sections we analyzed monotonicity of the decision rule. However, for empirical work it is the paths rather than the decision rule which can be observed. In this section we describe the connection between monotonicity of the decision rule and monotonicity of the optimal paths.

4.1 Comparative Statics

Even when the decision rule is nondecreasing, it is not necessarily true that an increase in the parameter θ will lead to a uniform increase of the optimal path. For example, when $\hat{s}^*(s,x;\theta) = \{s, s'\}$ with $s > s'$, for all (s,x,θ) the comparative statics when θ is varied are not necessarily monotone. Let $\theta > \theta'$ and note that the path $(s', s', \ldots)$ is optimal for θ while $(s, s, \ldots)$ is optimal for θ'. Therefore along any sample path, for all $t > 0$ $s^t(\theta) < s^{t\prime}(\theta')$. If the chosen optimal paths are $(s, s', s, s', \ldots)$ for θ and $(s', s, s', s, \ldots)$ for θ', then the ordering

[9] This is easy to see by setting $\epsilon = 1 - \delta$ and Taylor expanding for ϵ close to 0. This yields $k < (1-2\epsilon)/\epsilon^2$ and $k > (1-3\epsilon)/\epsilon^2$, a contradiction.

will reverse every period. Also, if the set $\hat{s}^*(s,x;\theta)$ contains any points that are unordered with respect to each other, then it can happen that for certain t, $s^t(\theta)$ and $s^{t\prime}(\theta')$ will be unordered.

However, if at each period the largest (or smallest) optimal action is chosen then the comparative statics are straightforward.

Corollary 1 *Assume that a DDP has a monotone decision rule (e.g. restricted full complementarities). Then for any initial conditions s^0, x^0 and any two parameter choices $\theta \geq \theta'$ for all $t > 0$ $\quad s^t(\theta) \geq s^t(\theta')$ along any sample path, where in every period the largest optimal action is always chosen. (This also implies that $s^t(\theta) \geq s^t(\theta')$ in the sense of first order stochastic dominance.)*

Proof: This follows directly from induction on the monotonicity of the decision rule and the fact that if $A \geq B$ in the strong set order, then $\max(A) \geq \max(B)$. $\square$

Thus there always exists a monotone selection. If we strengthen the assumptions of nondecreasing differences to (strictly) increasing differences, then any selection is monotone, as shown in [15].

Corollary 2 (Hopenhayn and Prescott [15]) *Assume that a DDP has restricted full strict complementarities.*[10] *Then for any initial conditions s^0, x^0 and any two parameter*

[10]Note that when an optimization problem has strictly increasing differences, then its optimum must be

choices $\theta \geq \theta'$ *for all* $t > 0$ $\quad s^t(\theta) \geq s^t(\theta')$ *along any sample path. (This also implies that* $s^t(\theta) \geq s^t(\theta')$ *in the sense of first order stochastic dominance.)*

For strict monotonicity it appears that significantly stronger assumptions are required, such as differentiability. See [9] for a discussion of strict monotonicity.

4.2 Time Monotonicity

In this section we study the monotonicity of the optimal path itself. In order to simplify the exposition we will only consider deterministic dynamic decision problems in this section and assume that whenever the decision rule is multi-valued, the largest element is chosen and thus implicitly assume that the decision rule is single valued.[11]

Definition 8

i) A path is "eventually time nondecreasing" (resp. "eventually time nonincreasing") if there exists a T *such that* $s^{t+1} \geq s^t$ *(resp.* $s^{t+1} \leq s^t$*) for all* $t \geq T$*.*

ii)A path is "eventually time monotonic" if it is either eventually time nondecreasing or eventually time nonincreasing.

iii) A path is "time nondecreasing" (resp. "time nonincreasing") if it is "eventually time

unique, except at a countable number of places. Thus if a parameter is divisible (e.g. in the interval $[0,1]$) then the optimal decision is single valued almost everywhere.

[11]Otherwise, it would be easy to choose nonmonotonic selections.

nondecreasing" (resp. "eventually time nonincreasing") for $T = 0$.

iv) A path is "time monotonic" if it is either time nondecreasing or time nonincreasing.

Both time monotonic and eventually time monotonic paths are extremely simple.

Theorem 6 *An optimal path for a dynamic decision problem with a monotone decision rule is eventually time monotonic if there exists a t such that either $s^t \geq s^{t+1}$ or $s^t \leq s^{t+1}$.*

The proof is immediate by induction.

Corollary 3 *An optimal path for a dynamic decision problem with a monotone decision rule is time monotonic if either $s^0 \geq s^1$ or $s^0 \leq s^1$.*

Another relatively simple possibility for the dynamics of a DDP is the following.

Definition 9

i)A path is "eventually time pseudo-nondecreasing" (resp. "eventually time pseudo-nonincreasing") if there exists some $T \geq 0$ and $r > 0$ such that $s^t \geq s^{t+r}$ (resp. $s^t \leq s^{t+r}$) for all $t \geq T$.

ii) A path is "eventually time pseudo-monotonic" if it is either eventually time pseudo-nondecreasing or eventually time pseudo-nonincreasing.

Theorem 7 *An optimal path for a dynamic decision problem with a monotone decision rule is eventually time pseudo-monotonic if there exist $t \geq 0$ and $r > 0$ such that either $s^t \geq s^{t+r}$ or $s^t \leq s^{t+r}$.*

When S is one-dimensional it is known [5] that supermodularity implies that the decision rule is monotone, and thus the optimal path is time monotone.

Somewhat surprisingly there exist DDPs with monotone decision rules which are not even time pseudo-monotonic, and thus the intuition from the one dimensional case does *not* carry over to higher dimensions. In fact there exist such DPPs with extremely complex optimal paths. Consider the following example.

Let $S = [0,1]^2$, $S(s) = S$, $\pi(s) = 0$ and

$$F(s,s') = -\frac{(s_1)^2}{2} - \frac{(s_2-1)^2}{2} + rs_1s_1's_2' + s_2 + rs_2(s_2'-1)(1-s_1').$$

It is easy to check that $F(s,s')$ has full complementarities.

Now consider the dynamics in the limit as $\delta \to 0$. In this case the optimal decision rule can be computed (from the first order conditions) as

$$s^*(s^{t-1}) = (\phi(rs_1's_2'), \phi(1 + r(s_2'-1)(1-s_1'))),$$

where $\phi(\gamma) = \gamma$ for $\gamma \in [0,1]$, $\phi(\gamma) = 1$ for $\gamma > 1$, and $\phi(\gamma) = 0$ for $\gamma < 0$.

Consider the case when $s_1^0 + s_2^0 = 1$. Note that $s_0^*(s_1^0, s_2^0) + s_1^*(s_1^0, s_2^0) = 1$ and thus along the optimal path $s_0^t + s_1^t = 1$ for all t. Thus the line $s_0 + s_1 = 1$ is invariant under the dynamics. Let $y^t = s_1^t - s_2^t$ describe the dynamics along this line. A straightforward computation yields (along the optimal path)

$$y^{t+1} = \phi(\frac{r}{2}(1 - {y^t}^2) - 1).$$

This equation is simply a transformed version of the logistic mapping. It is well known that for many values of r it has extremely complicated dynamics. In particular, when $r = 4$ the map is chaotic and leads to behavior that is (ergodically) indistinguishable from the flipping of a (biased) coin. We expect that such behavior would be preserved for δ small and positive; however, the direct proof of this statement is quite delicate, and we present below a simpler approach.[12]

Consider the case when $r = 3.839$ and the initial condition is $y^0 = .9186$. Then the optimal path is a 3-cycle (time pseudo-monotonic), $(a_1, a_2, a_3, a_1, a_2, a_3 \ldots)$ where (approximately) $a_1 = .9186$, $a_2 = -.7002$, and $a_3 = -.0216$. (This can be proven analytically, see e.g. [6], p.93.) It is easy to check that this 3-cycle is hyperbolic and therefore remains (although slightly shifted) under small changes to the dynamics. Thus, there exists some $\hat{\delta}$ such that for $0 < \delta \leq \hat{\delta}$ there exists a 3-cycle. However, this implies that there are many extremely complicated optimal paths for different initial conditions.

Theorem 8 (Friedman and Landsberg [12]) *If a nondecreasing mapping in $\Re^2$ has a 3-cycle then there exists topological chaos.*

This implies, for example, that there exist periodic paths of all periods, and aperiodic paths that come arbitrarily close to any periodic path. (See e.g. [4], for a discussion of

[12] Since the line $s_1 + s_2 = 1$ is a hyperbolic invariant manifold one should be able to show that the chaotic behavior is preserved, using the techniques in [14].

chaotic behavior in economics.) Note that the existence of chaos in this example is robust to small changes (in the C^2 metric) in F, since hyperbolic 3-cycles are structurally stable.

However, while the existence of chaotic paths in a monotone DDP is robust and thus there are many initial conditions that lead to extremely complex optimal paths, this does not imply that such paths arise often in a measure theoretic sense. It can be shown in the previous example that for almost all initial conditions the optimal path is *not* chaotic. In fact for any initial condition not on the line $s_1^0 + s_2^0 = 1$, the path is eventually monotone and converges to either $(0,0)$ or $(1,1)$.

In fact, it is conjectured [12] that except for a set of initial conditions of measure 0, all paths of a monotone mapping are eventually time pseudo-monotone.[13] Thus while the existence of complex paths is common in monotone mappings, for most initial conditions the optimal path will be well behaved.

Conjecture 1 *A DDP with a (single valued) monotone decision rule is eventually pseudo-monotonic for almost all $s^0 \in S$.*

When S is discrete (e.g. each S_i is countable) the conjecture is clearly true for all $s^0 \in S$.

[13]Hirsch [13] has proven that in the continuous time analog of a monotone DDP almost all paths are eventually monotone. However, this is not true for discrete time.

5 Complementarities in Real Options Models

In real options models (see, e.g. [8]) it is useful to write $F(s,s',x;\theta) = \pi(s,x;\theta) - C(s,s',x;\theta)$, where $\pi(s,x;\theta)$ represents the payoff for a single period with investment strategy s and exogenous prices x, while $C(s,s',x;\theta)$ is the cost of changing strategy from s' to s. Note that the costs may depend on the stochastic process, e.g. the cost of constructing a new manufacturing plant may depend on the price of labor or current interest rates. In this section we describe some interesting classes of submodular adjustment costs which can be useful in real options models. (Recall that $C(\cdot)$ is submodular if $-C(\cdot)$ is supermodular.)

Submodular costs are straightforward in the optimal stopping model, $S = \{0,1\}$. In this case the costs of transitions, $C(0,1)$ and $C(1,0)$, can have arbitrary non-negative (possibly infinite) values, and the problem will still have nondecreasing decision rules.[14] For example, note that when there are only two possible actions, fixed costs are submodular.

Consider the case where $S \subseteq \Re$ represents the amount of an asset being held. In this case there will often be proportional adjustment costs (such as a fixed percentage of the change, e.g. sales commissions) which can be easily included by setting $C(s,s') = \gamma|s-s'|$, where γ represents the percentage costs. More generally $C(s,s') = g(s-s')$ is submodular for any convex $g(\cdot) \geq 0$ such that $g(0) = 0$.

[14] The precise requirement is that $C(0,1) + C(1,0) \geq 0$. Otherwise, one could receive a positive payoff simply from switching back and forth repeatedly.

Although this can be proven directly from the definition of submodularity (as in [7]), it is much simpler to prove it in the differentiable setting by checking that $\frac{\partial^2 g(s,s')}{\partial s \partial s'} = -g''(\cdot) \leq 0$ from the definition of convexity. This extends to nondifferentiable functions since any convex function g can be approximated (pointwise) by smooth convex functions and submodularity is preserved under pointwise convergence. Also, if the set $S \subset \Re$ is discrete, then the previous argument shows that g is submodular on $\Re^2$ and thus must also be submodular on S^2 since it is a sublattice of $\Re^2$. Arguments such as these allows one to check submodularity using ordinary calculus, which is often convenient.

In real options models it is often useful to allow for the creation of new projects. In the complementarity based framework this is straightforward to model. Let S, which may be continuous, be the levels of operation for a potential factory where, if it were in operation, $\pi(s, x; \theta)$ is supermodular. Now suppose that the factory can be built at a cost of K dollars. Under the assumption that $\pi(s, x; \theta)$ is nondecreasing in x, θ we can easily include this in a model while preserving the monotonicity of the decision rule as follows. Define $S' = S \bigcup \{\gamma\}$ where $\gamma < s$ for all $s \in S$, and set $\pi'(s, x; \theta) = \pi(s, x; \theta)$ for $s \in S$ and $\pi'(\gamma, x; \theta) = 0$; therefore until the factory is built there is no payoff.

Theorem 9 *$\pi'(s', x; \theta)$ for $s' \in S'$ is supermodular in s' and has nondecreasing differences in x, θ.*

Proof: Supermodularity is immediate since γ is defined to be strictly less than any $s \in S$. Nondecreasing differences follow from considering the case with $s \in S$ and γ:

$$\pi'(s, x; \theta) - \pi'(\gamma, x; \theta) = \pi(s, x; \theta)$$

which is assumed to be nondecreasing in x, θ. $\Box$

The cost of construction can be included by defining $C(s, s') = K$ for $s \in S$ and $s' = \gamma$, and zero otherwise. Thus the combination of discrete changes with continuous variables is straightforward, which is a useful attribute of lattice theoretic methods.

We note a general property of submodular adjustment costs under the assumption that adjustment costs are zero when no change occurs, $C(s, s, x; \theta) = 0$, which we call the "small steps" property [10]. Chose any $s < s' < s'' \in S$, then $C(s, s') + C(s', s'') \leq C(s, s'')$ which implies that one large step is at least as costly as two small ones. In particular, this shows that if the action space has 3 or more elements, then fixed costs (e.g. $C(s, s') = 1$ if $s \neq s'$ and 0 otherwise), or any nonconvex costs, do *not* in general lead to robust monotone decision rules. Similarly, if $\pi(s, x; \theta)$ does not have nondecreasing differences then there exists some $\kappa(s)$ such that the decision rule is not nondecreasing everywhere.

Theorem 5 shows that if the complementarity assumptions in this section are violated, then there exists a set of functions $\kappa_i(s_i)$ such that the decision rule will not be monotone. However, the function κ_i could be quite complex, not reflecting any natural economic quantity. Fortunately, for reasonable models κ_i typically has a natural interpretation. We now present an example in which the class of functions κ_i is particularly simple.[15]

In this model a firm has production technology $z = f(s;\theta)$ where $s \in \Re^2$ are two types of physical capital, $\theta \in \Re$ is an exogenous parameter (such as the quality of the technology) and $z \in \Re^2$ are the outputs. Let $p = (p_1, p_2)$ be the price of the outputs, and δ the discount factor. There is a cost of adjusting physical capital and its depreciation rate is ν. Thus, in this model $\pi(s;\theta) = p_1 f_2(s;\theta) + p_2 f_2(s;\theta)$ and $C(s, s') = c_1(s'_1 - \nu s_1) + c_2(s'_2 - \nu s_2)$.

We make the following assumptions.

Assumption 3

i) $f(s;\theta)$ is continuously differentiable, concave, and nondecreasing in s, θ and $f(0;\theta) = 0$.

ii) $c_i(\cdot)$ is continuously differentiable, nondecreasing, and convex.

Now assume that there are costs incurred maintaining physical capital which are not known to the modeler. In order to show the necessity of complements for a monotone decision rule one only need consider positive linear maintenance costs.

Theorem 10 *The following statements are equivalent:*

i) For all i, $\frac{\partial^2 f_i(s;\theta)}{\partial s_1 \partial s_2} \geq 0$ and for all i, j, $\frac{\partial^2 f_i(s;\theta)}{\partial s_j \partial \theta} \geq 0$.

[15] The example follows one given by Amir [1] and details of it are discussed in [21].

ii) The optimal decision rule is nondecreasing in s, θ for all maintenance costs of the form $k_1 s_1 + k_2 s_2$ given $k_1, k_2 \geq 0$.

Proof: This is proved by combining Theorem 4 with a minor modification of Theorem 5 which is based on a result of Milgrom and Roberts [18] and is given as Lemma 1 in the Appendix. □

Note that the comparative statics in this model are simple.

Theorem 11 *If Condition (i) of the previous theorem is met, then the optimal path is nondecreasing in s^0, θ.*

Proof: This follows from Theorem 4. □

The results are similar to those in a recent model of Dixit [7], who analyzes investment and employment decisions when adjustments in both are costly. In our notation, $\pi(s, x; \theta)$ is the operating profit where $(s_1, s_2) \in S = \Re^2$ represent the capital and labor stocks respectively, while $\Theta = \Re$ represents the size of the exogenous shocks and $X = \Re$ is the stochastic process that generates the exogenous shocks which are $x^t\theta$. Dixit assumes that π is strictly concave in s, supermodular in (s, θ), and adjustment costs are convex.[16] Thus, as shown in the previous theorem, in order for his monotonicity results to be robust for any per period costs

[16] In particular, he chooses asymmetric and piecewise linear adjustment costs.

of labor and capital, π must be supermodular on its entire domain.[17] Thus, supermodularity is (robustly) necessary for his results.

Finally, we note that the effect of parameters on the adjustment costs is quite subtle. Let $S \subset \Re$ and consider an adjustment cost of the form $g(s - s'; \theta)$ where $\theta \in \Theta \subseteq \Re$ and g is convex in its first argument for fixed θ and smooth. Now, for g to have nondecreasing differences in $((s, s'), \theta)$ we must have both $\frac{\partial^2 g(s-s';\theta)}{\partial s \partial \theta} \geq 0$ and $\frac{\partial^2 g(s-s';\theta)}{\partial s' \partial \theta} \geq 0$. However, $\frac{\partial^2 g(s-s';\theta)}{\partial s \partial \theta} = -\frac{\partial^2 g(s-s';\theta)}{\partial s' \partial \theta}$ which implies that $\frac{\partial^2 g(s-s';\theta)}{\partial s \partial \theta} = -\frac{\partial^2 g(s-s';\theta)}{\partial s' \partial \theta} = 0$. Thus, in general one should not expect monotone statics for parameters related to the adjustment costs. However, it is possible in special cases.[18]

6 Appendix

In this appendix we prove the characterization theorem which is used in Theorem 5. Let $a = (s', x, \theta)$.

Theorem 12 *The following are equivalent:*

i) $f(s, a)$ has limited complementarities.

[17] Dixit [7] gives the following example: $\pi(s; \theta) = s_1^{\alpha} s_2^{\beta} \theta^{\gamma} - m s_1 - w s_2$ where $\alpha + \beta + \gamma = 1$. In this case robustness is equivalent to requiring monotonicity for all possible values of m, w.

[18] For example, [11] describes a model which has monotone statics for a parameter reflecting the size of the adjustment costs.

ii) For all $\kappa_i(s_i)$ $(1 \leq i \leq m)$, the solution

$$s(a) = argmax_{s \in S(a)} \left\{ f(s,a) + \sum_i \kappa_i(s_i) \right\}$$

is nondecreasing in x.

Proof: That (i)implies (ii) was shown in [19]. We now prove the converse. Given (ii) we consider the two possibilities.

I) $f(s,a)$ **does not have nondecreasing differences:** In this case we have $a, a' \in A$ with $a > a'$ and $\sigma > \sigma'$ which are both elements of $S(a) \bigcap S(a')$, such that $f(\sigma, a) - f(\sigma', a) < f(\sigma, a') - f(\sigma', a')$.

Given σ, σ' define the generated hypercube to be

$$HC(\sigma, \sigma') = \left\{ \nu \in S(a) \bigcap S(a') \; |\forall 1 \leq i \leq m, \; \nu_i \in \{\sigma_i, \sigma_i'\} \right\}.$$

Now we show that σ, σ' can be chosen so that $HC(\sigma, \sigma') = \{\sigma, \sigma'\}$. Assume that $\nu \in HC(\sigma, \sigma')$ but is not equal to either σ or σ'. Then

$$f(\sigma, a) - f(\sigma', a) = f(\sigma, a) - f(\nu, a) + f(\nu, a) - f(\sigma', a)$$

and similarly for a'. Thus either $f(\sigma, a) - f(\nu, a) < f(\sigma, a') - f(\nu, a')$ or $f(\nu, a) - f(\sigma', a) < f(\nu, a') - f(\sigma', a')$. Assume the former; then we can repeat this process with $HC(\sigma, \nu)$. Since $HC(\sigma, \sigma')$ is finite, we can continue this process until $|HC(\sigma, \sigma')| = 2$.

Assume that σ, σ' are chosen such that $|HC(\sigma,\sigma')| = 2$. Set $r = f(\sigma,a) - f(\sigma',a)$. Choose j such that $\sigma_j \neq \sigma'_j$. Now let $\kappa(s) = \sum_i \kappa_i(s_i)$ and choose $\kappa_i(s_i)$ equal to 0 for $s_i = \sigma_i$ (when $i \neq j$) and r when $i = j$ and $\kappa_i(s_i) = 0$ for $s_i = \sigma'_i$ and $\kappa_i(s_i) = -\infty$ (or some sufficiently negative number) otherwise. It follows by construction that $s^*(a) = \sigma'$ and $s^*(a') = \sigma$, where $s^*(\cdot)$ represents the optimal action. This violates monotonicity, since the only points that have payoffs which are not $-\infty$ are those in $HC(\sigma,\sigma')$ which contains only σ and σ' by construction.

II) $f(s,a)$ is not supermodular in s: By assumption for some $a \in A$ and $s' \in S$ there exist points $\sigma, \sigma' \in S(s',a)$ such that

$$f(\sigma,a) + f(\sigma',a) > f(\sigma \wedge \sigma', a) + f(\sigma \vee \sigma', a) \qquad (*).$$

In this case we can choose σ, σ' such that $|HC(\sigma \wedge \sigma', \sigma \vee \sigma')| = 4$. For example choose a $\nu \in HC(\sigma \wedge \sigma', \sigma \vee \sigma')$ such that $\nu \notin \{\sigma, \sigma', \sigma \wedge \sigma', \sigma \vee \sigma'\}$. Define $\alpha = \nu \wedge \sigma$, and $\beta = \nu \vee \sigma'$. Note that $\alpha \wedge \sigma' = \sigma \wedge \sigma'$, $\alpha \vee \sigma' = \beta$, $\sigma \vee \beta = \sigma \vee \sigma'$, and $\sigma \wedge \beta = \alpha$.

Rewriting (*)(and dropping the a) gives

$$0 > ([f(\sigma \vee \sigma') - f(\sigma)] - [f(\beta) - f(\alpha)]) + ([f(\beta) - f(\alpha)][f(\sigma) - f(\sigma \wedge \sigma')])$$

and at least one of these terms must be negative. If it is the first, then we see that

$$f(\sigma) + f(\beta) > f(\sigma \wedge \beta) + f(\sigma \vee \beta),$$

and

$$|HC(\sigma \wedge \beta, \sigma \vee \beta)| < |HC(\sigma \wedge \sigma', \sigma \vee \sigma')|.$$

Now assume that $|HC(\sigma \wedge \sigma', \sigma \vee \sigma')| = 4$. Choose j such that $\sigma_j \neq \sigma'_j$. For $i \neq j$ set $\kappa_i(s_i) = 0$ for s_i equal to σ_i or σ'_i and $-\infty$ otherwise. Set $\kappa_j(\sigma_j) = 0$, $\kappa_j(\sigma'_j) = f(\sigma, a) - f(\sigma', a)$, and $-\infty$ otherwise. For this problem, $f(\sigma, a) + \kappa(\sigma) = f(\sigma', a) + \kappa(\sigma')$ which implies that the set of optima (which is contained in $HC(\sigma \wedge \sigma', \sigma \vee \sigma')$) consists of σ, σ' but not both $\sigma \wedge \sigma'$ and $\sigma \vee \sigma'$, by the inequality above. This violates monotonicity, as a set which is not a sublattice is neither greater or less than itself in the strong set order. $\Box$

We note that the above procedure can be use to construct a κ which is constant everywhere except for a single neighborhood, which in the continuous case can be made arbitrarily small, thus providing a local characterization of supermodularity without requiring differentiability.

The following lemma is a simple modification of a result in [18].

Lemma 1 *Assume that $f(s, a)$ is continuously differentiable, concave and nondecreasing in s, a. Then the following are equivalent:*

i) $f(s, a)$ has full complementarities.

ii) For all $\kappa_i \in \Re_+$ $(1 \leq i \leq m)$, the solution

$$s(a) = argmax_{s \in S(a)} \left\{ f(s, a) + \sum_i \kappa_i s_i \right\}$$

is nondecreasing in x.

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