Strategic Properties of Heterogeneous Serial Cost Sharing

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Abstract

We show that serial cost sharing for heterogeneous goods [4], and a large number of other cost sharing mechanisms, have the same strong strategic properties as serial cost sharing for homogenous goods [10], including uniqueness of the Nash equilibrium for all utility profiles and cost functions, dominance solvability, solvability in overwhelmed actions, and robustness to coalitional deviations. We describe several applications to cost/surplus sharing and the Internet.

1 Introduction

In [10, 13], Moulin and Shenker introduced a cost sharing mechanisms with extremely strong strategic properties. Dubbed serial cost sharing (or fair share in the network context) this mechanism leads to games in which the Nash equilibrium is unique, robust to coalitional deviations and the only rationalizable strategy profile. In addition, this Nash equilibrium is the unique outcome of adaptive learning [7] and reasonable learning in asynchronous low information environments [5].

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The natural generalization of the serial mechanism to heterogeneous goods was introduced in [4]. In this note we show that this generalized serial mechanism maintains these same strategic properties. In addition, there is a large class of mechanisms which also share these strong strategic properties.

2 The Serial Mechanisms and Generalizations

Consider a typical cost sharing problem: there are n agents, each of whom has a demand $q_i \in Q_i = [0, q^{max}]$ for a good, where $i \in N = \{1, 2, \ldots, n\}$ and $q^{max} > 0$. (Note that we can have $q^{max} = \infty$.) The cost of serving these demands is $C(q)$, which must be divided among the agents; the cost share for agent i is given by $x_i(q; C)$.

Moulin and Shenker [10] considered the case where the goods are homogeneous i.e., the cost function can be written $C(|q|)$, where $|q| = \sum_{i=1}^{n} q_i$. They assume that $C(\cdot)$ is continuously differentiable and nondecreasing. Although serial cost is well defined for any such cost function, in order for it to have strong strategic properties, it is necessary that the cost function have increasing marginal costs. As in their paper, we will assume strictly increasing marginal costs to simplify the exposition.[1] This is quite natural in many network contexts for which serial cost (or fair share) is particularly applicable. A useful formulation of serial cost, for our purposes, is given by

$$x_i^{SC}(q; C) = \int_0^{q_i} \partial_i C(t) dF_i^{SC}(t; q)$$

where $F_i^{SC}(t; q) = (t/n)|\{j \in N \mid q_j \leq q_i\}|$ and $|S|$ is the cardinality of S.

[1] As they discuss, strictness can be relaxed by assuming that agents' preferences are strictly convex.

In [4], Friedman and Moulin extended serial cost to the case where the goods may be heterogeneous, i.e., $C(q)$ can be an arbitrary nondecreasing and continuously differentiable function of its n variables. They describe the cost sharing mechanism given by

$$x_i^{SC}(q; C) = \int_0^\infty \partial_i C(\gamma_i^{SC}(t; q)) d\gamma_i^{SC}(t; q) \qquad (*)$$

as the natural generalization of serial cost, where γ^{SC} is the path from 0 to q given by $\gamma^{SC}(t; q) = (tu) \wedge q$, where $(p \wedge q)_i = min[p_i, q_i]$ and u is the unit vector $(1, 1, \ldots, 1)$.

As we will see in Section 3, this heterogeneous version of serial cost will have the same strong strategic properties as the homogeneous version when the marginal cost is strictly increasing in all variables.

However, these strategic properties of heterogeneous serial cost are not unique to the serial mechanism, as they were in the homogeneous case (under symmetry and several other restrictions). In particular, consider any cost sharing mechanism which can be written as (*) where γ^{SC} is replaced by any continuous, non-decreasing path which can be written as follows. Let $\phi(t; C)$ be any function which is nondecreasing and continuous in t (for fixed C) with $\phi(0; C) = 0$ and $\lim_{t \to \infty} \phi(t; C) > q^{max}$ for any C, then let the path be defined by $\gamma^{SC}(t; q) = \phi(t; C) \wedge q$. We call these methods "fixed path methods" and note that they are analogous to the paths studied by Moulin in the discrete version of this problem in [8] and in both the continuous and discrete versions for a restricted class of homogeneous cost functions in [9].

For functions, ϕ, which are independent of C this includes many asymmetric methods, but the only symmetric one is the serial cost method. However, when ϕ depends nontrivially on

C there are many symmetric methods, since for any C, which is not symmetric for any group of agents, the notion of symmetry does not constrain the choice of a path at all. Clearly, such methods include many methods which could be ruled out by "locality assumptions" (i.e., $x(q;C)$ should not depend on properties of C outside of the domain $[0,q]$); however, there still exist symmetric methods which are local.

One interesting example is the Moulin-Shenker Ordinal method (discussed in [14]) for which the path $\phi(t;C)$ is the solution of

$$d\phi_i(t;C)/dt = 1/\partial_i C(t),$$

satisfying $\phi(0;C) = 0$, which in general is well defined only for cost functions for which $\partial_i C(t) > 0$ everywhere, but under our assumptions this is equivalent to requiring that $\partial_i C(0) > 0$. In fact, in this case, even this assumption can be dropped as there always exists at lease one solution to this equation with $\phi(0) = 0$ and when there are multiple such solutions, a particular one can be chosen in a variety of ways. Other simple, but perhaps uninteresting methods include $\phi_i(t;C) = \alpha_i(C)t$ where $\alpha_i(C)$ is any function of the derivatives of C at 0.

3 Cost Sharing Games

We now consider the strategic properties of cost sharing games when fixed path methods are used to allocate the costs. Following Moulin and Shenker [10] we assume that player i's utility is given by a concave utility function $U_i(q_i, x_i)$ which is increasing in q_i and decreasing in x_i.

Thus, given a set of players, N, a set of utility functions, U_i, a cost sharing method, $x(\cdot;\cdot)$ and a cost function C this defines a noncooperative game in strategic form. We now present several examples:

3.1 Examples

The original (and still highly relevant) example consists of a group of agents, who are Poisson sources of data sharing a network link.[2] In this case, the cost function is the expected queue size which is given by $C(q) = 1/(\mu - |q|)$ where μ is the capacity of the link. In this case serial cost was denoted fair share by Shenker [11] and was based on a protocol known as fair queuing [1].

When there are different types of traffic on the link the cost function is no longer homogeneous in the q_i's. In this case, the cost function is heterogeneous in the goods and once again we can apply any fixed path method to allocate the costs (queue lengths).

An immediate generalization (in the case of Poisson sources) arises when there is a complex network with agents located at different nodes. In this case, if serial cost (or any other fixed path method) is applied at each node, then the aggregate cost sharing corresponds to the application of heterogeneous serial cost to the aggregate cost function. This model is analyzed in [11].

There are a wide variety of queuing problems which are mathematically equivalent to this problem, such as the ordering of web server or database requests and the allocation of dial in modems.[3] They also arise in a variety of service facilities problems [6, 15, 3].

[2] See [5] for an extensive bibliography of such models.
[3] These are discussed in detail in [2].

In all of these queuing models, the marginal cost is typically increasing in all variables, due to stochastic congestion effects, and thus our results are applicable.

A different class of examples arise in surplus sharing problems where q_i it the effort of agent i and $C(q)$ is the group profit, when agents abilities are heterogeneous. In this case a player's utility is decreasing in q_i and increasing in x_i, but when $C(q)$ is concave and efforts are substitutes, our analysis is applicable.

4 Strategic Properties

We are interested in a variety of strategic properties which were analyzed by Moulin and Shenker [10]. These are: uniqueness of the Nash equilibrium, strong equilibria, uniqueness of the set of rationalizable equilibria and convergence of adaptive learners. In addition, we are interested in the convergence of reasonable learners in network settings [5].

We will show all of these results for fixed path methods by showing that the game is O-solvable [5]. These other properties all follow immediately as they are implied by O-solvability.

Definition 1

i) A strategy q_i is overwhelmed by q_i' with respect to set $S \subseteq Q_{-i}$ if

$$\max\{U_i(q_i, x_i(q; C)) \mid q_{-i} \in S_{-i}\} < \min\{U_i(q_i', x_i(q_i', q_{-i}; C)) \mid q_{-i} \in S_{-i}\}.$$

Let $O_i(S)$ denote the set of strategies which are not overwhelmed with respect to S_{-i}.

ii) A game is O-solvable if the set $O^\infty = \lim_{n\to\infty} O^n(Q)$ consists of a single strategy for each player, where O^n is the functional composition of O with itself n times.

We now present our main theorem.

Theorem 1 *Assume that the marginal cost ($\partial_i C(q)$) is strictly increasing in all variables, $x_i(\cdot;\cdot)$ is a fixed path method and that preferences, $U_i(q_i, x_i)$ are increasing in q_i, decreasing in x_i and concave. Then the induced game is O-solvable.*

The proof of this theorem is in the appendix. However, it is straightforward to show that this theorem implies all the other strategic properties of interest and more.[4]

Corollary 1 *Assume that the marginal cost is strictly increasing in all variables, $x_i(\cdot;\cdot)$ is a fixed path method and that preferences, $U_i(q_i, x_i)$ are increasing in q_i, decreasing in x_i and concave. Then the induced game:*

i) has a unique Nash equilibrium.

ii) has a unique rationalizable strategy.

iii) play will converge to the Nash equilibrium under adaptive learning.

iv) the unique Nash equilibrium is strong.

v)play will converge to the Nash equilibrium under reasonable learning in network settings.

Proof: Parts (i), (ii) and (iii) follow immediately from the fact that any overwhelmed strategy is also dominated and thus O-solvability implies dominance solvability which is well known to imply (i) and (ii) and was shown by Milgrom and Roberts [7] to imply (iii). Parts (iv)and (v) were proven in [5]. □

[4]In addition, O-solvability implies that the generalized Stackelberg equilibria of the game coincide with the unique Nash equilibrium and that when viewed as a revelation mechanism, the social choice function defined by a fixed path method is strictly coalitionally strategyproof and Maskin Monotonic. See, [5] for details.

Since serial cost for heterogeneous goods is a fixed path method this proves that heterogeneous serial cost maintains the same strategic properties as the homogeneous version.

Lastly, we note that Shenker [12] has studied the properties of abstract serial mechanisms in general production economies. The relationship between his abstract analysis and our results merits further study.

A Proof of Theorem 1

We first present some properties of fixed path methods. define $\lambda_i(q_i) = \min[t \mid \phi_i(t) \geq q_i]$.

Lemma 1 *Assume that marginal cost is strictly increasing in all variables and that* $x_i(\cdot;\cdot)$ *is a fixed path method. Define* $\lambda_i(q_i) = \min[t \mid \phi_i(t) \geq q_i]$*. Then:*

a) $x_i(q;C)$ *is strictly increasing and strictly convex in* q_i

b) $x_i(q;C)$ *is nondecreasing in* q_j *for all* $j \neq i$*.*

c) For all q *and* q'_j *such that both* $\lambda_j(q_j), \lambda_j(q'_j) \geq \lambda_i(q_i)$ *then* $x_i(q;C) = x_i(q_{-j}, q'_j;C)$*.*

Proof: Using this notation, we can write the cost sharing formula as

$$x_i^{SC}(q;C) = \int_0^{\lambda_i(q_i)} \partial_i C(\phi_i(t) \wedge q) d(\phi_i(t) \wedge q).$$

i) For any $\delta > 0$ let $\Delta_i(q) = x_i(q';C) - x_i(q;C)$, where $q' = (q_{-i}, q_i + \delta)$. First, note that

$$x_i^{SC}(q;C) = \int_0^{\lambda_i(q_i)} \partial_i C(\phi_i(t) \wedge q') d(\phi_i(t) \wedge q').$$

Thus

$$\Delta_i(q) = \int_{\lambda_i(q_i)}^{\lambda_i(q_i+\delta)} \partial_i C(\phi_i(t) \wedge q') d(\phi_i(t) \wedge q').$$

Since $\partial_i C(p) > 0$ for $p > 0$, $\Delta_i(q) > 0$, $x_i(q; C)$ is strictly increasing in q_i. Also, since $\partial_i C(p)$ is strictly increasing in p, $\Delta_i(q)$ is strictly increasing in q_i and thus $x_i(q; C)$ is strictly convex in q_i, completing the proof of (a).

ii) For any $\delta > 0$ and $j \neq i$ let $\eta_i(q) = x_i(\hat{q}; C) - x_i(q; C)$, where $\hat{q} = (q_{-j}, q_j + \delta)$. Since the paths $\gamma(t; q)$ and $\gamma(t; q')$ agree for $t < \lambda_j(q_j)$, if $\lambda_j(q_j) \geq \lambda_i(q_i)$, then $\eta_i(q) = 0$, proving (c). Assume that $\lambda_j(q_j) < \lambda_i(q_i)$ then

$$\eta_i(q) = \int_{\lambda_j(q_j)}^{\lambda_i(q_i)} \left\{ \partial_i C(\phi_i(t) \wedge q') d(\phi_i(t) \wedge q') - \partial_i C(\phi_i(t) \wedge q) d(\phi_i(t) \wedge q) \right\}.$$

Since $(\phi_i(t) \wedge q') > (\phi_i(t) \wedge q)$ for $t > \lambda_j(q_j)$

$$\eta_i(q) > \int_{\lambda_j(q_j)}^{\lambda_i(q_i)} \left\{ \partial_i C(\phi_i(t) \wedge q) d(\phi_i(t) \wedge q') - \partial_i C(\phi_i(t) \wedge q) d(\phi_i(t) \wedge q) \right\}.$$

Let $\gamma = \partial_i C(\phi_i(t') \wedge q) - \partial_i C(\phi_i(t') \wedge q)$ where $t' = (\lambda_j(q_j) + \lambda_i(q_i))/2$. Then

$$\eta_i(q) > \int_{t'}^{\lambda_i(q_i)} \gamma d(\phi_i(t) \wedge q') > 0,$$

proving (b). $\square$

The following lemma is a slight modification of Theorem 10 in [5].

Lemma 2 *Consider a game $< G, Q >$, where G is the payoff function and Q_i is a compact subset of $\Re_+$. Assume there exist a set of nondecreasing functions $\lambda_i : Q_i \to \Re_+$ such that the following hold:*

i) $G_i(q) \geq G_i(\tilde{q}_j, q_{-j})$ for any $\tilde{q}_j \geq q_j$, $i \neq j$.

ii) $G_i(q_j, q_{-j}) = G_i(\tilde{q}_j, q_{-j})$ for any $q_j, \tilde{q}_j$ such that both $\lambda_j(q_j), \lambda_j(\tilde{q}_j) \geq \lambda_i(q_i)$, $\forall i \neq j$.

iii) for each q_{-i} there exists an element $BR_i(q_{-i})$ such that

$$x_i \neq BR_i(q_{-i}) \Rightarrow G_i(BR_i(q_{-i}), q_{-i}) > G_i(x_i, q_{-i}).$$

iv) $BR_i(q_{-i}) = BR_i(\tilde{q}_j, q_{-ij})$ *for any* $\tilde{q}_j$ *such that* $\lambda_j(\tilde{q}_j) \geq \lambda_i(BR_i(q_{-i}))$.

Then the game is O-solvable.

The proof of this lemma is a straightforward modification of the proof (of Theorem 10) given in [5], which is for the case in which $\lambda_i(q_i) = q_i$.

We now show that games arising from generalized path methods satisfy these conditions:

i) This follows from (b) since $x_i(q; C)$ is nondecreasing in q_j.

ii) This follows directly from (c).

iii) The uniqueness of the best reply follows from the strict convexity of $x_i(q; C)$ w.r.t. q_i and the convexity of preferences.

iv) The best reply occurs at the unique point of tangency between the indifference curves and the opportunity set, in the (q_i, x_i) plane. Neither of these is (locally) affected by such changes in q_j by (c).

Thus, the game induced by a fixed path method satisfies the assumptions of the lemma and must be O-solvable, completing the proof. □

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