Optimization Based Characterizations of Cost Sharing Methods

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Abstract

We provide several new characterizations of well known cost sharing methods (CSMs) as maxima of linear (or convex) functionals. For the Shapley-Shubik method the characterization has an interpretation in terms of randomly ordered agents choosing their most preferred CSM, while the characterizations of the Aumann-Shapley and Serial methods have a very general character: any symmetric convex functional which uniquely characterizes a scale invariant CSM must characterize the Aumann-Shapley method, while the identical statement is true for the Serial method when scale invariance is replaced by demand monotonicity.

1 Introduction

The problem of allocating costs among a group of agents is an important problem that arises in a wide variety of economic (and noneconomic) settings.[1] Since the are many ways to share costs, it is important to distinguish between various cost sharing methods (CSMs). The most common methods of comparing CSMs ares based either on normative criteria or strategic criteria.[2] In this paper, we introduce a third, which has aspects of both – optimization

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[1]See, e.g., [13, 9] for a surveys and extensive bibliographies.

[2]Obviously, there is a great deal of overlap between these two. See [4] for further discussion on this topic.

criteria.

We consider the characterization of a CSM as the optimal solution of a continuous functional over a subset of the set of all CSMs. While this problem is generally intractable over the set of all CSMs, surprisingly, it is often tractable over the set of additive CSMs (the class most commonly studied) and specific further subsets of interest, such as the scale invariant or demand monotonic CSMs. In particular, these problems become directly solvable when the functional is convex and the problem is one of maximization. In that case, the solution set must contain an extreme point, which has a very simple representation as a continuous monotone path.

Using this representation, it is often straightforward to prove that a CSM is optimal for a certain functional. We give examples of interesting functionals and prove the completeness of the characterizations. We also discuss practical methods to solve particular examples and then use this approach to provide optimization based characterizations of three important CSMs: the Shapley-Shubik, Aumann-Shapley and Serial methods.

Our characterization of the Shapley-Shubik Method has the following interpretation: assume that all goods are of the same type (complements or substitutes) and that agents are ordered randomly. The agents are then asked, in some fixed order, to choose their optimal CSM based on their own subjective beliefs about what the cost function will be. If the first agent's choice is unique, then that method is chosen, if not, then the next agent gets to choose her optima from the set of optima of the first. This procedure the continues through all the remaining agents. Theorem 1 states that the outcome of this procedure for any order will be the incremental method with that order, and thus when we average over all orders this produces the Shapley-Shubik method.[3]

Our characterizations of the Aumann-Shapley and Serial methods have a "universal character." In these, we consider the characterizations arising from symmetric functionals over the set of scale invariant of demand monotonic methods. Theorem 2 says that if a symmetric (and convex) functional characterizes unique scale invariant method then that method must be the Aumann-Shapley method. Theorem 3 is identical for the Serial method when scale invariance is replaced by demand monotonicity. Thus, if we are interested in symmetric characterizations, it is not possible to characterize any scale invariant method *besides* the Aumann-Shapley method.

2 Optimization over the Additive Cost Sharing Methods

Let $N = \{1, 2, \ldots, n\}$ be the set of agents. Each agent's demand is $q_i \in \Re_+$ and the cost of serving these demands is $C(q)$ with $C \in \mathcal{C}$, where $\mathcal{C}$ is the set of nondecreasing, continuously differentiable functions from $\Re^n_+$ to $\Re_+$, satisfying $C(\vec{0}) = 0$.[4] A cost sharing mechanism provides a method for computing the cost shares allocated to each of the agents. An additive cost sharing method (CSM) is defined as follows:

Definition 1 *An additive cost sharing method is a mapping, $x : \Re^n_+ \times \mathcal{C} \rightarrow \Re^N_+$, satisfying:*

1) Efficiency: $\sum_{i \in N} x_i(q; C) = C(q)$,

[3]Note the similarities between this characterization of that for the Shapley method, in which agents are also randomly ordered, served in that order and charged their marginal costs, according to the order [12].

[4]Note that the generic vector of 0's will be denoted by $\vec{0}$ and the unit vector of 1's by $\vec{1}$, where the dimension of these vectors will be obvious from the context.

2) Additivity: for all $C, D \in \mathcal{C}$ and all $i \in N$ the following holds: $x_i(q; C + D) = x_i(q; C) + x_i(q; D)$,

3) Dummy: For any $C \in \mathcal{C}$ and $i \in N$ such that $\partial_i C(p) = 0$ $\forall p \in \Re^n_+$, then $x_i(q; C) = 0$ for all $q \in \Re^n_+$.

Let CS denote the set of all such CSMs. Note that we use the notation $\partial_i C(p)$ to represent the partial derivative of $C(q)$ with respect to q_i evaluated at p. The set CS has several useful properties:

Proposition 1 ([3]) *The space CS is convex and compact in the topology of weak convergence.*

We now present our main tool, a representation theorem of CS, based on the path methods.

Definition 2 *A path function γ is a mapping $\gamma : \Re_+ \times \Re^n_+ \rightarrow \Re^n_+$ satisfying the following for each $q \in \Re^n_+$:*

1) $\gamma(t; q)$ is continuous and nondecreasing in t.

2) $\gamma(0; q) = \vec{0}$ and there exists a $\hat{t} > 0$ such that for all $t \geq \hat{t}$, $\gamma(t; q) = q$.

Let the set of all such path functions be denoted Γ. Also, for each $q \in \Re^n_+$, let $\Gamma(q)$ be the projection of Γ onto its second component, for fixed q. Given a path function, it is straightforward to construct its related CSM.

Definition 3 *Given a path $\gamma \in \Gamma$ the "path method generated by γ" is given by*

$$x_i^\gamma(q; C) = \int_{t=0}^{\infty} \partial_i C(\gamma(t; q)) d\gamma_i(t; q).$$

As proven in [5], a path method is a valid CSM. It is shown in [3] that any CSM can be constructed as a convex combination of path methods. We will require only the following weaker result:

Proposition 2 ([3]) *$x \in ext(CS(q))$ if and only if $x = x^\gamma$ for some $\gamma \in \Gamma(q)$.*

Now, consider some optimization problem:

$$\max_{x(q;\cdot) \in CS(q)} V[x(q; \cdot)] \qquad \text{(OPT(V))}$$

where V is convex. Our main tool is the following lemma about this optimization problem.

Lemma 1 *Assume that V is a convex function on CS and let $MAX(V)$ be the set of solutions to $OPT(V)$. Then:*

i) $MAX(V)$ is nonempty and contains a path method.

ii) If V is affine then $MAX(V)$ is convex and the extreme points of $MAX(V)$ are path methods.

iii)If V is strictly convex then $MAX(V)$ contains only path methods.

Proof: Since V is convex it must be continuous and thus existence follows from the compactness of CS. Standard arguments imply statement (ii) and (iii). $\diamond$

Note that this result only applies for fixed q; however, with additional constraints we can extend this result to the case with varying q.

2.1 Scale invariance and demand monotonicity

First, consider scale invariance, which is well known and was used in the classic axiomatization of the Aumann-Shapley method [2, 7].

Given $\lambda \in \Re^N_{++}$, define $\tau_\lambda(q)$ by $\tau_\lambda(q)_i = \lambda_i q_i$ for $i \in N$ and define $\tau_\lambda(C)$ by $\tau_\lambda(C)(q) = C(\tau_\lambda(q))$, for all $C \in \mathcal{C}$.

Definition 4 (Scale Invariance) *A CSM, $x \in CS$, is scale invariant if $x(\tau_\lambda(q); C) = x(q; \tau_\lambda(C))$, for all $\lambda \in \Re^n_{++}$ and $C \in \mathcal{C}$.*

We will also need a technical assumption:[5]

Definition 5 *A CSM, $x \in CS$, is* continuous at 0 *if for all $C \in \mathcal{C}$, $i, j \in N$ and $q \in \Re^n_+$, $\lim_{q'_i \to 0} x_j(q_{-i}, q'_i; C) = x_j(q_{-i}, 0; C)$.*

When combined with continuity at 0, the set of scale invariant CSMs are generated by the scale invariant paths. Define Γ_{SI} to be the set of paths such that $\gamma(t; q) = \tau_q(\gamma(t; \vec{1}))$. Thus each component is $\gamma_i(t; q) = \gamma_i(t; \vec{1}) q_i$. Let SI be the set of CSMs which are scale invariant and continuous at 0. The characterization of this set was given in [3]. For our purposes the following result will suffice.

Proposition 3 ([3]) *The space SI is convex and $x \in ext(SI)$ if and only if $x = x^\gamma$ for some $\gamma \in \Gamma_{SI}$.*

Another important axiom is demand monotonicity [8, 5]. For technical reasons, we only consider the case of bounded demands when studying demand monotonic CSMs. Thus,

[5] See [5] for a discussion on the need for this assumption and the changes that arise without it.

whenever we discuss demand monotonicity we require that there exists some $0 < \overline{q} < \infty$ and that $q \in [0, \overline{q}]^n$.

Definition 6 (Demand Monotonicity) *A CSM in CS is demand monotonic if for all $q, q' \in [0, \overline{q}]^n$ such that $q_i \leq q'_i$ and $q_{-i} = q'_{-i}$ and all $C \in \mathcal{C}$: $x_i(q; C) \leq x_i(q'; C)$.*

In this case, we also need a somewhat stronger technical assumption:[6]

Definition 7 *A CSM $x \in CS$ is* dummy invariant *if for all $C \in \mathcal{C}$ such that agent $i \in N$ is a dummy agent ($\partial_i C \equiv 0$), $x_j(q; C)$ is independent of q_i for all $j \in N$.*

Now we define the paths which will characterize the demand monotonic and dummy invariant CSMs, denoted by DM. The extreme CSMs are constructed from a single path. Let $\alpha(t)$ be a nondecreasing path defined on $t \in \Re_+$ with $\alpha(0) = \vec{0}$ such that there exists a $\hat{t} > 0$ such that for all $t > \hat{t}$, $\alpha(t) \geq \overline{q}$. Let Γ_{DM} be the set of all such paths. Given $\alpha \in \Gamma_{DM}$ define γ^α by $\gamma^\alpha(t, q)_i = \min[\alpha_n(t), q_i]$, for all $i \in N$.

Proposition 4 ([3]) *For any $\overline{q} < \infty$, the space DM is convex and $x \in ext(DM)$ if and only if $x = x^\gamma$ for some $\gamma \in \Gamma_{DM}$.*

Now, we have the analogous results to Lemma 1 for both scale invariant and demand monotonic CSMs.

Consider some optimization problem:

$$\max_{x(\cdot;\cdot) \in Y} V[x(\cdot; \cdot)] \qquad (OPT_Y(V))$$

where V is convex, and Y is either SI or DM.

[6] This is a fixed population version of dummy consistency, introduced in [3].

Lemma 2 *Assume that V is a convex functional on Y where Y is either SI or DM and let $MAX_Y(V)$ be the set of solutions to $OPT_Y(V)$, Then $MAX_Y(V)$ is nonempty and contains a path method, x^γ, with $\gamma \in \Gamma_Y$.*

3 Examples and Completeness of Characterizations

In this section we provide some economic motivation for the optimization of convex functionals as a method of characterizing a CSM. Then we show that within this class it is possible to characterize any CSM, thus showing that the characterization is complete, even when restricted to an interesting class of functionals.

Consider a social planner who must choose a CSM to maximize some social welfare function $W(x(q;C))$. The social planner knows the demands, q, but is uncertain about the actual cost function that will arise. (Alternatively, assume that the CSM must be chosen in advance and then used for a large number of cost functions.) The social planner has some social welfare function for allocations which can be written $W(x(q;C))$. Given beliefs β over $\mathcal{C}$ the social planner's problem would be to maximize $V(x) = E_C[W(x(q;C) \mid \beta]$.

For example, one possible preference for the planner could be to minimize agent 1's payments, $W(x(q;C)) = x_1(q;C)$, which might be the case if agent 1 were the planner. Similarly, the planner could maximize any weighted sum of expected payments, $W(x(q;C)) = \sum_{i \in N} a_i x_i(q;C)$. Note that in these cases V is linear and hence convex. Note also, that in this case, the welfare function simplifies since x is assumed to be linear, $V(x) = E_C[W(x(q;C)] = W(x(q;E[C]))$.

Thus, in this simple case, the social planner's problem is

$$\max_{x \in CS(q)} \sum_{i \in N} a_i x_i(q; E[C]),$$

which can be simplified, using Lemma 1 to

$$\max_{\gamma \in \Gamma(q)} \sum_{i \in N} a_i x_i^{\gamma}(q; E[C]),$$

which is equal to the following

$$\max_{\gamma} \sum_{i \in N} \int_0^1 a_i \partial_i E[C](\gamma(t; q)) d\gamma_i(t; q),$$

where $\gamma(0; q) = 0$, $\gamma(1; q) = q$, $\gamma(t; q)$ is nondecreasing and continuous in t. This is a problem of optimal control and can be solved using standard techniques [6].

We can extend this to the case where both q and C are uncertain. In this case, if we want to choose $x \in CS$ then the problem separates into one problem for each q, while if we restrict to scale invariant (or demand monotonic) CSMs then using Lemma 2 we can reduce the problem to a single optimal control problem of the type just discussed.

Another example arises from demand monotonicity. The main justification of demand monotonicity is as an incentive constraint; demand monotonicity helps deter agents from artificially inflating their demands. Thus, we might be interested in finding the "most demand monotonic CSM," e.g., $W(x(q; C)) = \sum_{i \in N} a_i \partial_i^{\epsilon} x_i(q; C)$, where we use the "finite partial derivative" $\partial_i^{\epsilon}(f(q)) = [f(q) - f(q_i(1-\epsilon), q_{-i})]/(q_i \epsilon)$ since the true derivative may not exist. One could then maximize this over the set of demand monotonic CSMs, or even over the set of scale invariant CSMs.

Lastly even in the limited set of functionals discussed above, it is possible to characterize (almost) any CSM in $ext(CS(q))$. Given a continuously differentiable path $\gamma \in \Gamma(q)$ define $B(p) = d(\gamma, p)^2$ where $d(\gamma, p) = \min_t ||\gamma(t; q) - p||$, the Euclidean distance from p to the path γ. The function $B(p)$ is continuously differentiable, non-negative, and is equal to 0 if p is on the path γ. Now define $C(p) = \int_0^{p_1} B(t, p_{-1}) dt$.[7] Let $V(x) = -x_1(q; C)$. Then the $x \in CS(q)$ that maximizes this is by Lemma 1 a path method, and it is easy to see that the original path maximizes the functional, since $\partial_i C(p) = B(p)$ which is 0 along the path and strictly positive anywhere else. Thus, we have uniquely characterized γ using a simple linear functional. Note that this argument easily extends to the common case where the path is continuously differentiable except at a finite set of t's. Also, note that any path is the pointwise limit of such paths and thus this argument can be used to approximate any path to any degree of accuracy.

4 A Characterization of the Shapley-Shubik Method

In this section we provide a new characterization of the Shapley-Shubik method, which is simply the application of the Shapley value to the cost sharing setting when the demands are considered fixed. This characterization is an optimization based variant of a standard interpretation of the Shapley value for cooperative games.[8] The standard interpretation is

[7] Formally, $C(p)$ may not be monotone and therefore not an element of $\mathcal{C}$; however, we can simply add a term of the form $\sum_{i \in N} \lambda(p_i + p_i^2)$ where λ is chosen sufficiently large to guarantee that the sum is nondecreasing and the apply dummy and additivity to show that this additional piece does not change the following argument.

[8] Another characterization of the Shapley-Shubik method is given in [5]. In that paper the Shapley-Shubik method is shown to be the unique method satisfying Dummy, Additivity, Scale Invariance, Demand Monotonicity and Symmetry.

that the agents arrive in random order, state their demands and are charged their cost, which is the additional cost of serving their demand, conditional upon all the previous demands. The new characterization is similarly based on having agents arrive randomly, but in this case, the cost function is not yet known, and the agent requests the set of possible CSMs based on her beliefs over the set of possible cost functions which may arise. If we allow these set of beliefs to be arbitrary, except for the assumption that all goods complements, then the expected costs of this procedure are precisely those given by the Shapley-Shubik formula.[9] (This result is unchanged if all agents believe that goods are substitutes instead of complements.)

Let $\mathcal{C}^+$ be the subset of $\mathcal{C}$, consisting of cost functions C for which all goods are strict complements:[10] for all $i \in N$ and $p \in \Re^n_+$, $\partial_i C(p)$ is strictly increasing in p_{-i}. Now assume that each agent, has beliefs β_i about the cost function which will be chosen from $\mathcal{C}^+$ and that q_i, the demand for each agent, is common knowledge. Each agent wants to minimize their (subjective) expected payments $V_i(x) = E[x_i(q;C)|\beta_i]$.

Consider an ordering of the agents, $\psi \in \Psi$, where ψ is a bijection from N to $\{1, 2, \ldots, n\}$. The Incremental method with order ψ, x^ψ is given by $x_i^\psi(q;C) = C(q_{S(\psi,i)}, 0_{-S(\psi,i)}) - C(q_{S(\psi,i)\setminus i}, 0_{-S(\psi,i)\setminus i})$, where $S(\psi, i) = \{j \in N \mid \psi(j) \leq \psi(i)\}$. Note that the Incremental method with order ψ is generated by the path, γ^ψ, given by the path which connects the points, $\gamma^\psi(i;q) = (q_{S(\psi,i)}, 0_{-S(\psi,i)})$ with straight lines in increasing order. We will now show

[9] In some sense this is perhaps a more direct extension of the Shapley Value to cost sharing when compared to the standard extension in which each infinitesimal unit of demand is viewed as an agent, which leads to the Aumann-Shapley method.

[10] The result also holds for weak complements with the addition of a nondegeneracy condition.

that the procedure described above leads to the Incremental method.

For fixed $q \in \Re^n_+$, define $ARGMAX(V, B)$ to be the set of maximizers of V from the set B. Now define $B^\psi_1 = ARGMAX(V_{\psi(1)}, CS(q))$ and inductively define $B^\psi_k = ARGMAX(V_{\psi(k)}, B_{k-1})$ for $1 < k \leq n$.

Theorem 1 *For any $\psi \in \Psi$, beliefs β_i, and $q \in \Re^n_+$, the set B_n consists of the Incremental method with order ψ.*

Proof: For notational simplicity consider the ordering $\psi(i) = i$. First consider $ARGMAX(V_1, CS(q))$. This extreme points of this set consist of path methods by Lemma 1, and for any of these path methods,

$$V_1(x^\gamma) = \int_0^\infty \partial_1 C(\gamma(t; q) d\gamma_1(t; q).$$

Now, it is easy to see that if a path γ does not first go along the line from 0 to $(q_1, 0_{-1})$ then it leads to a higher payment then that path that does, since $\partial_1 C(p)$ is strictly increasing in p_{-1}. Thus, B_1 is the convex hull of all paths of this type. A similar argument then shows that B_2 is the convex hull of paths which go from 0 to $(q_1, 0_{-1})$ and then to $(q_1, q_2, 0_{-\{1,2\}})$. Proceeding inductively proves the result. $\square$

Recall that the Shapley-Shubik method is given by $x^{SS} = (n!)^{-1} \sum_{\psi \in \Psi} x^\psi$. Then the characterization of the Shapley-Shubik method is obtained by averaging over all possible orderings.

Corollary 1 *Assume that $\psi \in \Psi$ is chosen randomly, with uniform probability. For any beliefs β_i, and $q \in \Re^n_+$, the expected payment of agent i, conditional on any realization of*

$C \in \mathcal{C}$, is given by $x^{AS}(q;C)$, when the method is chosen with the same procedure as in Theorem 1.

Note that the result is unchanged when the set $\mathcal{C}^+$ is replaced by the set $\mathcal{C}^-$, the set of all cost functions for which all goods are strict substitutes. Also, this procedure can be elegantly summarized as the maximazation of a lexicographical social choice function, for which the agents are exogenously (but randomly) ordered.

5 Universal Characterizations of the Aumann-Shapley and Serial Methods

In this section we provide "universal" characterizations of the Aumann-Shapley and Serial methods. We will show that for any symmetric and convex functional on the space of scale invariant (resp. demand monotonic) CSMs that have a unique maximizer, that this maximizer must be the Aumann-Shapley (resp. Serial) method. Thus, any strict symmetric characterization over scale invariant (resp. demand monotonic) CSMs can only yield the Aumann-Shapley (resp. Serial) method.

We now describe this result formally. First we recall that the Aumann-Shapley method [1], x^{AS}, is generated by the path $\gamma(t;q) = tq$ while the Serial method [10, 5], x^{SER}, is generated by the path given by $\gamma_i(t;q) = \min[t, q_i]$. Let $x \in Y$ where $Y \in \{SI, DM\}$ and for any permutation of N, $\psi \in \Psi(N)$, define $\psi(x)$ by $\psi(x)(q;C) = x(\psi(q);\psi^{-1}(C))$ where $\psi(q)_i = q_{\psi(i)}$ and $\psi^{-1}(C)(p) = C(\psi^{-1}(p))$.

Definition 8 *A functional V on Y where $Y \in \{SI, DM\}$ is symmetric if for all $\psi \in \Psi(N)$*

and $x \in Y$, $V(\psi(x)) = V(x)$

Thus, a functional V is symmetric if it is invariant to a relabeling of the agents. Now we present our result:

Theorem 2 *If V is symmetric and $|MAX_Y(V)| = 1$ for $Y \in \{SI, DM\}$, then $MAX_Y(V)$ is either the Aumann-Shapley method, x^{AS}, (if $Y = SI$) or the Serial method, x^{SER}, (if $Y = DM$).*

Proof: By Lemma 2, since $|MAX_Y(V)| = 1$ we must have that $MAX_Y(V) = x^\gamma$ with $\gamma \in \Gamma_Y$. By assumption, $V(\psi(x^\gamma)) = V(x^\gamma)$, but note that $\psi(x^\gamma) = x^{\psi(\gamma)}$ where $\psi(\gamma)(t; q)_i = \gamma_{\psi(i)}(t; \psi(q))$ and thus we must have $\psi(x^\gamma) = x^\gamma$. For $Y = SI$, assume that $q = \vec{1}$, then the only way this equality can hold is if $\gamma(\cdot; \vec{1}) = \psi(\gamma)(\cdot; \vec{1})$ which is only true when $\gamma(\cdot, \vec{1})$ is the straight line from $\vec{0}$ to $\vec{1}$. By scale invariance, this restricts the path to be, in general, the straight line from $\vec{0}$ to q, which is the path that generates the Aumann-Shapley method. When $Y = DM$ the analogous argument holds by applying symmetry to the cases when q is on the line $t \mapsto t\vec{1}$. This uniquely determines α as the that line, $\alpha(t) = t\vec{1}$ (up to a monotone transformation of t), which generates the Serial method. □

6 Concluding Comments

Characterizing a CSM as the maximum of a functional, V, is an extremely general procedure and can encompass normative approaches, where V is social welfare function to be maximized, strategic, where V is the indicator function for certain strategic requirements, or even traditional approaches from mechanism design, where e.g., V is the principal's profit arising from a certain CSM. However, our analysis does not encompass these methods as we are restricted to additive CSMs and convex functionals; nonetheless, our analysis does apply to some interesting functionals and provides a new perspective on characterization problems.

The procedure we propose can be viewed either from Rawlsian perspective [11] in which the social planner is behind a veil of ignorance about the cost function that will arise, or a more regulatory perspective in which the regulator must choose the CSM before the specific details of a project are known. This can also model the situation in which a group of firms, or people, determine the process of sharing costs of a joint project before the details are known.

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